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PLANE-WAVE TIME-DOMAIN ALGORITHMS FOR EFFICIENT ANALYSIS OF THREE-DIMENSIONAL TRANSIENT WAVE PHENOMENA

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THESIS

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ABSTRACT

Computer simulation of linear transient wave phenomena involving structures that reside in an unbounded medium to generate broadband data is of paramount importance in such disciplines as acoustics, electromagnetics, and geophysics. Numerical techniques to perform such simulations call for the evaluation of retarded-time boundary integrals (RTBIs) as these boundary integrals play a fundamental role in formulation of the integral equation-based numerical techniques and in imposing exact radiation boundary conditions used in differential equation-based methods. However, evaluation of RTBIs using classical techniques is a computationally expensive procedure limiting the applicability of numerical transient analysis techniques to a small number of problems. In this thesis, numerical schemes for analyzing transient wave interactions with large-scale structures are introduced. Central to these techniques are the formulation of integral equations that are capable of producing stable and accurate results and the development of plane-wave time-domain (PWTD) algorithms for efficient evaluation of RTBIs. In the first part of this thesis, time domain combined field integral equations for analyzing acoustic and electromagnetic wave scattering are introduced and their superiority, in terms of accuracy and stability, to currently used integral equations is shown. In the second part, two PWTD algorithms based on Whittaker and finite-cone plane wave expansions of radiated fields are introduced. It is shown that the computational complexity of evaluating RTBIs can be considerably reduced by using these PWTD algorithms within a two-level or a multilevel framework. The last part of the thesis demonstrates the efficacy of using the resulting two-level and multilevel schemes both in integral equation-based transient scattering analysis from impenetrable obstacles and in imposing exact boundary conditions in the finite-difference time-domain technique. Numerous acoustic and electromagnetic examples are presented that show that the introduced schemes bring the transient analysis of wave scattering from realistic large-scale structures within the reach of current computational resources.
To my mother and my father
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CHAPTER 1
INTRODUCTION

1.1 Background

With the recent need for efficient wide-band system characterization, the scientific community has expressed a renewed interest in the analysis of short-pulse radiation and transient wave scattering phenomena [1-9]. Analysis of transient linear wave phenomena is of paramount importance in a wide range of disciplines including acoustics, electromagnetics, elastodynamics, and geophysics. Efficient computational analysis of these phenomena hinges upon the availability of fast time domain algorithms.

Present numerical transient analysis techniques are based on either differential equations or integral equations. The differential equation–based techniques such as the finite-difference time-domain (FDTD) technique [10, 11] and the time-domain finite-element method [12, 13] rely on a volumetric discretization of the fields that exist in and around the structure under consideration. Therefore, they are more suitable for analyzing wave interactions with inhomogeneous structures. These techniques become more favorable if the domain of interest is surrounded by boundaries characterized by simple local boundary conditions such as the Dirichlet and Neumann boundary conditions. If, on the other hand, the structure of interest resides in a homogeneous unbounded medium, schemes for emulating the Sommerfeld radiation condition on a finite truncation boundary must be devised. A boundary condition to rigorously fulfill this task can be derived from Huygens’ equivalence principle, and has been known by the name “exact boundary condition” for a number of years [14-18]. However, the exact boundary condition has not gained popularity due to its exorbitant computational requirements associated with the evaluation of retarded-time boundary-integrals (RTBIs) which form the backbone of its formulation. Instead, approximate boundary conditions based on one-way wave equation [19-
22] and, recently, perfectly matched absorbing layers [23, 24] have found widespread interest as mesh truncation methods.

On the other hand, integral equation–based techniques solve for a set of equivalent sources, which reproduce the field scattered by the structure under consideration when they radiate in free space [25-27]. Based on either the volume or surface equivalence principles [28, 29], these methods automatically impose the Sommerfeld radiation condition and call for a discretization of equivalent sources that occupy no more space than the original scatterer. This proves to be a great advantage when the scatterer is homogeneous or impenetrable because in these cases the problem can be formulated only in terms of equivalent sources that reside on the surface of the scatterer, thus reducing the dimensionality of the problem by one. However, time domain integral equation (TDIE) techniques, also known as marching-on-in-time (MOT) schemes [3, 4, 25, 30, 31], have long been conceived as intrinsically unstable and computationally expensive when compared to their differential equation counterparts. More than a decade of studies of instabilities associated with these techniques have yielded some understanding toward their origins and produced schemes to suppress or avoid them [3, 4, 31-47]. Of particular interest is the relation of instabilities to using implicit time stepping and accurate evaluation of RTBIs that appear in these methods. First noted by Barkeshli et al. [48] and later inspected empirically by Dodson et al. [43], implicitness and accurate approximations seem to play a key role in obtaining schemes that are stable for all practical purposes. However, as will be demonstrated later in this thesis, these two elements may not be sufficient to tame any instability, and additional measures might be necessary to obtain stable as well as accurate results.

In contrast, literature on rigorous techniques for reducing the high computational complexity of TDIE techniques is scarce at best. This is in spite of the fact that the last decade has witnessed significant speed-up of frequency domain integral equation solvers with the advent of the fast multipole method (FMM) [49-53], the impedance matrix localization technique [54], the multilevel matrix decomposition algorithm [55], etc. The culprit responsible for the high
computational cost of TDIE techniques is the evaluation of an RTBI as was the case for the exact boundary condition. Simply put, an RTBI relates the field to its sources that are distributed over a surface or a boundary. Due to the finite speed of propagation, the effects of sources residing at different points on the boundary contribute to the field at a point at distinct "retarded" times. In a typical scenario, the surface sources of temporal extent $T$ can be represented in terms of $N_s \propto S_A (\omega_{\text{max}} / c)^2$ spatial and $N_t \propto T \omega_{\text{max}}$ temporal samples, where $S_A$ denotes the area of the surface, $\omega_{\text{max}}$ is the temporal bandwidth of the source signatures, and $c$ is the wave speed in the medium. Evaluating the field at a single point due to this source distribution via an RTBI requires $O(N_t N_s)$ operations. In both the integral equation–based schemes and the exact boundary condition, fields need to be evaluated either on or in the vicinity of the surface at $O(N_s)$ points, thus incurring a total computational complexity of $O(N_t N_s^2)$. Recently, a technique for reducing this cost, albeit at the expense of accuracy, was suggested by Dodson et al. [56]. In this approach, integration over regions of low source values is omitted and field values are computed only in regions where the field is anticipated to be significant. However, cost savings only seem attainable when the scatterer is long compared to the width of the incident pulse along its direction of propagation, and an a priori estimate of the incurred error in the evaluated field values seems hard to obtain. Other preliminary research [57] has indicated that fast methods, similar in spirit to the frequency domain algorithms mentioned above, can also be developed in the time domain.

1.2 Statement of Purpose

The purpose of this thesis is to develop novel, fast, and rigorous time domain algorithms for evaluating RTBIs and demonstrate their use and effectiveness in reducing the computational complexities associated with analyzing transient interactions of acoustic and electromagnetic waves with structures residing in unbounded media. In the frequency domain, these goals have been achieved by FMM, which relies on expressing radiated fields in terms of a plane wave basis [49-51, 53, 58, 59]. However, this algorithm cannot be converted to the time domain in a
straightforward manner due to analytic properties of the terms that appear in the required expansions [60-67] and the presence of an anticausal “ghost” signal in time-domain plane-wave expansions [2, 6]. The plane-wave time-domain (PWTD) algorithms introduced in this thesis circumvent these difficulties by relying on propagating plane wave expansions of radiated fields derived entirely in the time domain [2, 6] and by exploiting the properties of the ghost signal to time-gate it out. In this dissertation, algorithms based either on a Whittaker-type field expansion [2, 68, 69], which relies on a complete spherical expansion of the far fields, or on a finite-cone representation [6] will be presented. The resulting algorithms permit the rapid evaluation of transient fields produced by source distributions that are of finite support in space and time. It is shown that the fields can be calculated to desired (and preset) accuracy by sacrificing from speed using these algorithms. In order to apply the PWTD algorithms within a general scenario, two-level and multilevel schemes are introduced and it is shown that the cost of evaluating RTBIs using these schemes scale as low as $O(N_t N_s^{4/3} \log N_s)$ and $O(N_t N_s \log N_s)$, respectively. Furthermore, how existing RTBI evaluation codes (within the context of MOT solvers and exact boundary conditions) can be retrofitted with these schemes is also elucidated. With these low complexity schemes, analysis of transient scattering from acoustical and electromagnetic targets of unprecedented sizes becomes feasible. However, the solutions to present TDIEs for analyzing such phenomena may be corrupted by spurious oscillations due to numerical excitation of interior resonance modes. Hence, leveraging off the vast body of literature that addresses the same issue in the frequency domain, TDIEs that do not support internal modes have to be formulated before embarking on large-scale scattering analysis. Time domain combined field integral equations, which do not support any interior modes, for analyzing scattering from impenetrable structures are also formulated in this thesis.

1.3 Organization of Chapters

The core of this dissertation consists of five chapters. Each chapter is also broken up into two main sections. In Chapter 2, which introduces relevant integral equations and RTBIs, and in
Chapter 6, which explains efficient implementation of exact boundary conditions, the two main sections discuss these issues for the acoustic and electromagnetic cases, separately. This way, a fundamental concept regarding linear wave phenomena is presented with applications to two different fields without losing focus. However, the two sections in each of these chapters are sufficiently decoupled so that the reader interested in only one of the applications can skip the other with no loss. This is not so in the other three chapters (Chapters 3-5) as the second part of these chapters expand upon—and therefore heavily rely on—the ideas introduced in the preceding parts. Chapter 3 introduces the theory underlying the PWTD algorithms that rely on Whittaker-type (nonwindowed) and finite-cone (windowed) field representations with diagonal translation operators for the simplest possible RTBI characterizing wave propagation. Two-level and multilevel PWTD enhanced MOT schemes are also outlined for each representation in this section. Chapter 4 demonstrates the use of the nonwindowed two-level and multilevel PWTD enhanced MOT schemes in solving the time domain combined field integral equation for the analysis of acoustic scattering from large-scale rigid bodies introduced in the first part of Chapter 2. Similarly, Chapter 5 outlines the use of the nonwindowed two-level and multilevel schemes within the MOT solution of the time domain combined field integral equation for the analysis of transient electromagnetic scattering from perfect electrically conducting bodies. Again, to make the discussions regarding acoustics and electromagnetics self-contained, Chapters 4 and 5 are decoupled from each other (although each make extensive use of the ideas presented in Chapter 3 and the relevant part of Chapter 2). Synopses of the individual chapters are as follows.

In Chapter 2, integral equations for tackling large-scale scattering problems involving impenetrable structures are developed for acoustics and electromagnetics. For the acoustic case, it is numerically demonstrated that the solutions to the Kirchhoff integral equation and its normal derivative, which are commonly used for transient analysis, can be corrupted by spurious oscillations. The source of this corruption is identified as the excitation of modes of the cavity formed by the surface of the scatterer by numerical inaccuracies. Inspired by the Burton-Miller
approach of eliminating interior resonant modes in the frequency domain solutions to integral equations [70], a time domain combined field integral equation (CFIE) is formulated. Invoking conservation of energy arguments, it is shown that the proposed CFIE provides an intrinsic mechanism to dissipate any interior fields—a property not shared with the Kirchhoff integral equation or its normal derivative. Similarly, for the electromagnetic case, the deficiency of the time domain electric field and magnetic field integral equations in producing accurate results due to internal resonance problems is illustrated and a CFIE that is free from this deficiency is formulated. A variety of numerical experiments demonstrating the effectiveness of using the proposed CFIEs are also presented in this chapter.

In the first part of Chapter 3, a three-stage nonwindowed PWTD algorithm based on a Whittaker-type expansion for the rapid evaluation of fields that satisfy the scalar wave equation is introduced. In the three stages of the algorithm, the time dependent source distribution is mapped onto plane waves (or rays) emanating from the source region; these plane waves are translated into rays that impinge onto the observation region; and the observed fields are reconstructed by projecting the impinging plane waves onto observer locations. These three stages are reminiscent of the aggregation, translation, and disaggregation steps of the frequency domain FMM. After experimentally verifying the accuracy of the algorithm, two-level and multilevel schemes to incorporate it into existing RTBI evaluation codes are also prescribed in this section. It is shown that the computational cost of performing a scattering analysis using the nonwindowed two-level and multilevel schemes scale as \( O(N_pN_s^{1.5} \log N_s) \) and \( O(N_pN_s \log^2 N_s) \), respectively. The second part of Chapter 3 expands on the ideas introduced in the first part to arrive at algorithms with even lower computational complexities. In this section, a finite-cone representation of radiated fields—a generalization of the Whittaker-type expansion—is presented. In particular, it is demonstrated that to reconstruct the observed fields it is sufficient to translate plane waves whose propagation directions fall within a cone that encloses the observation region instead of translating rays that propagate in all directions. It is
also shown that the translation functions associated with the finite-cone representation need to have shorter temporal extents, which is achieved by truncation of the translation function spectrum with a smooth window. Validity of the proposed PWTD algorithm and its exponential convergence characteristics are also demonstrated by numerical experiments in this chapter. The chapter concludes by outlining windowed two-level and multilevel PWTD enhanced RTBI evaluation algorithms, whose computational complexities are derived to be \( O(N_t N_s^{4/3} \log N_s) \) and \( O(N_t N_s \log N_s) \), respectively.

Chapter 4 describes how the two-level and multilevel nonwindowed PWTD enhanced MOT schemes can be adapted to speed up the MOT solution of the acoustic CFIE introduced in Chapter 2. To this end, first the plane wave expansion of the fields given by the RTBI related to the acoustic CFIE is presented. Then, implementation details for constructing a three-stage algorithm out of the presented plane wave expansion are outlined. Finally, the two-level and multilevel PWTD-enhanced MOT schemes are described in detail and their efficacy demonstrated via numerous examples. It is experimentally shown that the proposed methods become more advantageous to employ in comparison to the classical methods when the number of spatial unknowns \( N_s \) exceeds approximately 1600. It is also shown that use of the two-level and multilevel schemes helps increase the size of tractable problems from a few thousand unknowns to tens of thousands and over a hundred thousand spatial unknowns, respectively.

Chapter 5 accomplishes the task performed in the previous chapter—namely, implementation of the nonwindowed two-level and multilevel schemes in MOT solvers for acoustics—in electromagnetics. The vector nature of the electromagnetic phenomena permits further savings in the PWTD algorithm without affecting the overall complexity via the following observations. The propagating plane waves describe the far field radiation pattern of a given source distribution and therefore are characterized completely by the components of the field that are transverse to the propagation direction. Hence, the observed field can be reconstructed by translating only two-component vector fields rather than applying the PWTD
procedure to each component of the three-component field. For electromagnetic scattering analysis, the proposed schemes are shown to become more advantageous to use when the number of spatial unknowns $N_s$ exceeds approximately 1000 and scaling of the tractable problem size is similar to the acoustic case.

Chapter 6 accomplishes a slightly different task than the preceding two chapters. Application of the nonwindowed multilevel scheme in efficiently imposing exact boundary conditions for truncating FDTD domains is discussed both for the acoustic and electromagnetic cases. It is shown through numerical experiments that the targeted computational complexities are attained with reasonable accuracy constraints.

Finally, Chapter 7 summarizes the thesis and discusses possibilities for future research.
CHAPTER 2
TIME DOMAIN COMBINED FIELD INTEGRAL EQUATIONS

2.1 Introduction

Although MOT schemes for solving retarded time integral equations, e.g. the Kirchhoff integral equation, date back to 1962 [30], two important deficiencies have prevented them from attaining popularity: late time instabilities and high computational complexity. Especially due to the high computational cost of these schemes, they were not employed in analyzing transient scattering from objects whose dimensions span several wavelengths within the frequency range of interest. However, as mentioned in the previous chapter, with the recent advances, faster and more stable MOT schemes that can be used in the analysis of large-scale problems are now becoming available. This, in turn, brings forth the never before encountered deficiencies of these methods. One such deficiency, namely the corruption of the numerical solutions to retarded time integral equations with the resonant modes of the corresponding interior problem, is inspected in this chapter. In particular, time domain integral equations used in the analysis of acoustic scattering from rigid bodies and electromagnetic scattering from perfect electrically conducting (PEC) scatterers are considered.

In the acoustic case, the Kirchhoff integral equation (KIE) [71] and its normal derivative (NKIE) [72] have been widely used. Similarly, the time domain electric field and magnetic field integral equations (EFIE and MFIE) have been used in the transient electromagnetic analyses. It is well known that the frequency domain counterparts of all these integral equations suffer from the nonuniqueness difficulty—also known as the interior resonance problem [70, 73-76]. Briefly, the interior resonance problem is due to the fact that the resonant modes of the cavity
formed by the scatterer's surface lie within the null space of the linear operator that relates the equivalent surface sources to the scattered field. Therefore, at the resonance frequencies of the corresponding interior cavity problem the solutions to the exterior scattering problem are nonunique. A rigorous proof of this fact can be found in [70] and [74] among other references. In practice, this difficulty leads to ill-conditioned matrices when the frequency of interest approaches to one of the resonance frequencies of the interior problem. Several approaches have been proposed to overcome the nonuniqueness difficulty in the frequency domain. The approach that seems to have received wide acceptance in both the acoustical and electromagnetics communities is to use a linear combination of the two integral equations that can be used in the formulation of the same problem. In acoustics, Burton and Miller [70] have shown that a linear combination of the Helmholtz integral equation with its normal derivative is devoid of the nonuniqueness difficulty. Mautz and Harrington [74] presented a similar proof of uniqueness for a linear combination of frequency domain EFIE and MFIE in electromagnetics and called the resulting integral equation the combined field integral equation (CFIE).

At first glance, one would not expect the resonant modes of the corresponding interior problem to produce any difficulties in solving the basic time domain integral equations (i.e., KIE, NKIE, MFIE, and EFIE) since the initial conditions guarantee the uniqueness of the solutions to these equations. However, as will be demonstrated via numerical experiments in this chapter, the solutions to these equations are often corrupted by resonant modes when the incident field spectrum encompasses one or more of the resonance frequencies of the associated interior problem. In the next section, previous work on the effects of interior resonances in the MOT solutions to the retarded time integral equations is reviewed. Through this review, it becomes clear that two factors contribute to the corruption of numerical solutions by resonant modes. The first factor is the ability of the underlying integral equation to support any resonant fields. In theory, whether or not the integral equation being solved can sustain internal fields seems to be
irrelevant since no internal fields should be excited by a pulse incident from the outside [77]. This is where the second factor comes in: In the numerical solution, due to the inaccuracies introduced in the discretization, fields do couple into the cavity formed by the scatterer’s surface, albeit at low levels. Since little can be done in completely eliminating the numerical excitation of interior fields, the remedy to the interior resonance problem in the time domain seems to lie in formulating integral equations that do not support any interior fields. Inspired by the research in the frequency domain, whether the time domain CFIEs for analyzing acoustical and electromagnetic wave scattering have this property is inspected in Sections 2.3 and 2.4, respectively. It should be stressed that this time the aim is not to show that the solutions to the proposed CFIEs are unique but to prove that these integral equations do not sustain any internal fields. To this end, arguments based on the conservation of energy are invoked, and it is shown that the proposed CFIEs intrinsically dissipate any interior fields and therefore suppress any non-physical oscillatory components in their solution. Although, the arguments can be considered heuristic, much supporting evidence is provided within the numerical results presented in each section.

2.2 Background

Earlier implementations of the MOT schemes that relied on explicit time stepping for solving the basic time domain integral equations were often reported to yield late time instabilities [3, 4, 31, 32, 34]. Naturally, considerable research effort has been directed towards improving our understanding of the source of these instabilities [31-36, 39-44, 46, 47, 78]. In this regard, the contribution of interior cavity modes to instabilities has been investigated both theoretically and through numerical experiments [34, 35]. In these studies, the time domain integral equation is considered as a linear time invariant system that is characterized by the location of its transfer function poles—often referred to as the singularity expansion method.
(SEM) poles [79]. The poles corresponding to the resonant modes reside on the imaginary axis and, theoretically, are never excited by an externally incident field [77]. However, in a numerical solution procedure, the coupling between the incident field and the internal resonance modes does not vanish exactly, and “all the resonant solutions are likely to be excited to some extent by the incident field” [34]. Rynne and Smith [34] also state that “the inaccuracies induced by the numerical discretization of the integral equation causes some SEM poles, which theoretically should lie on the imaginary axis, to move into the right half plane.” Then, the corresponding components of the solution grow exponentially due to the positive real part of the pole, giving rise to instabilities. The poles with larger imaginary parts have a better chance of drifting into the right half plane since the loss of accuracy is more pronounced at higher frequencies. To prevent the instabilities due to the shifting of these high-frequency poles, time averaging procedures have been suggested [34, 38]. Indeed, in applications where the scatterer’s resonance frequencies were beyond the frequency range of interest, the averaging process that “filtered out” the unwanted frequency components proved useful in enforcing stability with negligible loss of accuracy [38]. Later, it was observed that the highly oscillatory instabilities could be eliminated by using implicit MOT schemes that are characterized by larger time step sizes than those used in explicit methods [41, 43, 45, 48]. This observation is in line with Rynne and Smith’s work since resonant solutions with frequencies exceeding \(1/(2\Delta_T)\), where \(\Delta_T\) is the time step size, cannot be represented at all by the temporal discretization. Hence, for sufficiently large \(\Delta_T\), only the low-frequency modes, which are less likely to result in instabilities, can be adequately represented by the discretization. Both of these approaches eliminate the highly oscillatory instabilities. However, if some poles of the system that fall within the frequency range of interest are excited, then the solution may still be corrupted by spurious oscillations. It should be noted that these poles are less likely to result in instabilities since they are not affected by numerical inaccuracies as much as the high frequency poles. Clearly, the effect of these
oscillations on the solution is highly dependent on the details of the implementation, as they are the byproduct of numerical inaccuracies.

2.3 Time Domain Integral Equations for Acoustics

In this section, the problems stemming from the presence of resonant modes in the transient analysis of acoustic wave scattering from closed rigid bodies are studied, and a Burton-Miller type CFIE is proposed as a remedy. Section 2.3.1 introduces the problem to be solved. Derivations of the Kirchhoff integral equation and its normal derivative are reviewed in Sections 2.3.2 and 2.3.3, respectively, and a Burton-Miller-type time domain CFIE to be used towards obtaining a numerical solution to this problem is introduced in Section 2.3.4. Energy arguments are invoked to determine whether these integral equations can sustain internal fields in Section 2.3.5. Section 2.3.6 outlines the formulation of the MOT method used in this study to solve the aforementioned integral equations. Numerical results are presented in Section 2.3.7, respectively. Details regarding the numerical implementation of the MOT method are discussed in Section 2.3.8.

2.3.1 Description of the acoustic scattering problem

Consider a rigid body residing in a homogeneous medium with wave speed $c$ that is illuminated by an incident field with velocity potential $\varphi^i(r, t)$ (Figure 2.1). Let $S$ denote the surface of the body, and let $\hat{n}$ be the position-dependent, outward pointing unit normal to $S$. Hypothetical surfaces conformal to $S$, but residing just inside and just outside $S$, are denoted by $S_-$ and $S_+$, respectively. When $\varphi^i(r, t)$ interacts with $S$, a scattered field with potential $\varphi^s(r, t)$ is generated such that the total velocity potential $\varphi^t(r, t) = \varphi^i(r, t) + \varphi^s(r, t)$ satisfies the boundary condition $\hat{n} \cdot \nabla \varphi^t(r, t) = 0$ on $S$. It is assumed that no interaction between the
incident field and the scatterer takes place prior to \( t = 0 \); hence, \( \varphi^i(r, t) = 0 \) on \( S \) for \( t < 0 \). Using the Kirchhoff-Helmholtz integral theorem [80], \( \varphi^s(r, t) \) can be expressed as

\[
\varphi^s(r, t) = \int_S \text{d}r' \varphi^i(r', t) \ast \mathbf{n}' \cdot \mathbf{n}' \cdot \nabla' \frac{\delta(t - R/c)}{4\pi R},
\]

(2.1)

where \( \ast \) denotes temporal convolution, \( \delta(\cdot) \) is the Dirac impulse, and \( R = |r - r'| \). Hence, once \( \varphi^i(r, t) \) on \( S \) is found the scattered field—and hence the fields everywhere—can be calculated by using Eq. (2.1). Next, three different integral equations that permit the reconstruction of \( \varphi^i(r, t) \) on \( S \) are considered.

![Figure 2.1: Description of the acoustic surface scattering problem.](image)

### 2.3.2 Kirchhoff integral equation

The incident, scattered, and total pressure fields are related to their respective velocity potentials as

\[
\rho_0 \varepsilon^{\xi}(r, t) = -\rho_0 \partial_t \varphi^{\xi}(r, t),
\]

(2.2)

where \( \xi \in \{t, i, s\} \) and \( \rho_0 \) is the fluid density of the ambient medium. The Kirchhoff integral equation [71] is obtained by setting the total pressure field on \( S_- \) to zero, i.e.,

\[
p^i(r, t) = -p^s(r, t) \quad r \in S_-.
\]

(2.3)
Using Eqs. (2.1) and (2.2) in Eq. (2.3) and taking its limit as \( r \) approaches a smooth part of \( S \) yields an integral equation that relates the unknown surface potential \( \varphi^i(r,t) \) to the incident field as

\[
-\partial_t \varphi^i(r,t) = -\frac{1}{2} \partial_t \varphi^i(r,t) + \partial_t \int_S d\mathbf{r}' \varphi^i(r',t) \cdot \hat{n}' \cdot \nabla' \frac{\delta(t-R/c)}{4\pi R} \quad \text{for} \quad r \in S, \quad (2.4)
\]

where \( L_p \) is the pressure field RTBI operator and the integral is to be evaluated in the principal value sense. Note that, assuming that no initial conditions on \( \varphi^i(r,t) \) are imposed, there exist many \( \varphi^i(r,t) \neq 0 \) that satisfy the homogenous equation \( L_p \{ \varphi^i(r,t) \} = 0 \). These fields are superpositions of the resonant modes of a soft cavity bounded by the surface \( S \) [70]. However, these solutions to Eq. (2.4) are ruled out since they do not satisfy the condition that \( \varphi^i(r,t) = 0 \) on \( S \) for \( t < 0 \).

### 2.3.3 Normal derivative of the Kirchhoff integral equation

The incident, scattered, and total velocity fields are related to their respective velocity potentials as

\[
\mathbf{v}^\xi(r,t) = \nabla \varphi^\xi(r,t), \quad (2.5)
\]

where \( \xi \in \{t,i,s\} \). The normal derivative of the Kirchhoff integral equation [72] is obtained by setting the normal component of the velocity field on \( S \) to zero, i.e.,

\[
\hat{n} \cdot \mathbf{v}^i(r,t) = -\hat{n} \cdot \mathbf{v}^s(r,t) \quad \text{for} \quad r \in S. \quad (2.6)
\]

Note that Eq. (2.6) is also valid on \( S_- \) and \( S_+ \) because the normal component of the velocity field is continuous across \( S \). Using Eqs. (2.1) and (2.5) in Eq. (2.6) yields an integral equation which relates the unknown surface potential to the incident field as
\[ \hat{n} \cdot \nabla \varphi'(r, t) = -\int_S dr' \varphi'(r', t) \ast \hat{n} \cdot \nabla \hat{n}' \cdot \nabla' \delta(t - R/c) \frac{\delta(t - R/c)}{4\pi R} \quad r \in S, \]  
\[ \L_v \{ \varphi'(r, t) \} \]

where \( \L_v \) is the velocity field RTBI operator and the integral is to be evaluated in the finite part sense \[81\]. Again, there exist many \( \varphi'(r, t) \neq 0 \) that satisfy the homogenous equation \( \L_v \{ \varphi'(r, t) \} = 0 \); these fields are the superpositions of the resonant modes of a hard cavity bounded by \( S \) \[70\]. However, again, these solutions to Eq. (2.7) do not satisfy the initial conditions and therefore should be ruled out.

2.3.4 Burton-Miller–type time domain combined field integral equation

As mentioned in Subsections 2.3.2 and 2.3.3, although both Eqs. (2.4) and (2.7) permit internal resonances, the condition that \( \varphi'(r, t) = 0 \) on \( S \) for \( t < 0 \) rules out any resonant components in the solutions to these equations. However, due to numerical inaccuracies introduced in the discretization of these integral equations, the resonant modes of the scatterer (or slightly perturbed versions thereof) are often excited, and the solutions to these equations are therefore corrupted by the presence of spurious oscillations. Numerical solutions to the frequency domain counterparts of Eqs. (2.4) and (2.7) (Helmholtz integral equation and its normal derivative) are also corrupted in the vicinity of the frequencies at which the scatterer’s resonant modes render the solution nonunique. Burton and Miller have shown that a linear combination of the Helmholtz integral equation and its normal derivative does not support any internal cavity modes, thereby guaranteeing a unique solution \[70\]. Here, we introduce a similar time domain CFIE. The mechanism by which this approach eliminates the corruption of the solutions by internal modes will be discussed in the next subsection.

The Burton-Miller–type time domain CFIE is derived by imposing a linear combination of the conditions expressed by Eqs. (2.3) and (2.6) as
\[(1 - \alpha)\partial_t \varphi^i(r, t) + \alpha c \hat{n} \cdot \nabla \varphi^i(r, t) = -(1 - \alpha)\mathcal{L}_p\{\varphi^i(r, t)\} + \alpha c \mathcal{L}_v\{\varphi^i(r, t)\} \quad r \in S, \quad (2.8)\]

where $\alpha$ is a real constant whose range is to be determined. Using Eqs. (2.1), (2.2), and (2.5) in Eq. (2.8), and letting $r$ approach $S$, yields the integral equation

\[(1 - \alpha)\partial_t \varphi^i(r, t) + \alpha c \hat{n} \cdot \nabla \varphi^i(r, t) = -(1 - \alpha)\mathcal{L}_p\{\varphi^i(r, t)\} + \alpha c \mathcal{L}_v\{\varphi^i(r, t)\} \quad r \in S. \quad (2.9)\]

Note that the combined field RTBI operator $\mathcal{L}_c$ is a linear combination of the pressure and velocity field operators $\mathcal{L}_p$ and $\mathcal{L}_v$, and that Eq. (2.9) reduces to Eqs. (2.4) and (2.7) for $\alpha = 0$ and $\alpha = 1$, respectively.

2.3.5 Corruption of numerical solutions to time domain integral equations by internal fields

The interpretation of time domain integral equations as linear time invariant systems and the relation of the resonant modes to the SEM poles of these systems were already alluded to in Section 2.2. It was also mentioned that the internal modes of the scatterer are always excited to some extent due to numerical inaccuracies. Here, we present the same picture from a different perspective and invoke energy arguments to study whether the integral equations (2.4), (2.7), and (2.9) sustain internal fields.

Assume that at time $t = t_0$ a spurious source distribution that supports resonant modes exists on $S$ because of the numerical excitation of (perturbed) resonance poles. Of course, this source distribution generates a field in the cavity bounded by $S_-$. The law of conservation of energy dictates that [80]
\[ W(t) = W(t_0) - \int_{t_0}^{t} dt' \int_{S_\text{-}} dr' \hat{n} \cdot \left[ p'^{(r', t')} v'^{(r', t')} \right], \]  

(2.10)

where

\[ W(t) = \frac{1}{2} \rho_0 \int_{V_\text{-}} dr' \left\{ |v'^{(r', t')}|^2 + \left[ p'^{(r', t'}/(c \rho_0) \right]^2 \right\} \]  

(2.11)

is the energy stored in volume \( V_\text{-} \) enclosed by \( S_\text{-} \) at time \( t \). If the Kirchhoff integral equation is used, \( p'^{(r, t)} = 0 \) on \( S_\text{-} \) is automatically enforced by Eq. (2.3). On the other hand, if the normal derivative of the Kirchhoff integral equation is used, \( \hat{n} \cdot v'^{(r, t)} = 0 \) on \( S_\text{-} \) by Eq. (2.6) and by the continuity of \( \hat{n} \cdot v'^{(r, t)} \) across \( S \). In either case, the integral over \( S_\text{-} \) in Eq. (2.10) equals zero and the energy that has leaked into the cavity is trapped there as \( W(t) = W(t_0) > 0 \) for all \( t \geq t_0 \). In other words, Eqs. (2.4) and (2.7) do not provide an \textit{intrinsic} mechanism for reducing the energy inside the cavity. However, if the combined field formulation is used, Eqs. (2.8) and (2.10) imply that

\[ W(t) = W(t_0) - \frac{(1-\alpha)}{\alpha} \frac{1}{c \rho_0} \int_{t_0}^{t} dt' \int_{S_\text{-}} dr' \left[ p'^{(r', t')} \right]^2. \]  

(2.12)

For \( W(t_0) > 0 \), \( \left[ p'^{(r', t')} \right]^2 = \left[ \hat{n} \cdot v'^{(r', t')} c \rho_0 \alpha/(1-\alpha) \right]^2 \) on \( S_\text{-} \) cannot be zero for all \( t' \geq t_0 \), since the energy inside the cavity will at some point interact with the cavity walls. During this interaction, the integrand in Eq. (2.12) will be positive and the energy stored in the cavity will decrease if \( 0 < \alpha < 1 \). This process will go on until \( W(t) = 0 \). Hence, for \( 0 < \alpha < 1 \), Eq. (2.9) provides an intrinsic mechanism for reducing the energy present inside the cavity.

While the above insights into the behavior of the introduced integral equations are heuristic in nature, much supporting evidence for the resonance-suppressing character of the CFIE comes from the numerical experiments reported in Section 2.3.7. Our results indicate that the CFIE
does indeed provide an intrinsic mechanism that suppresses all energy that (numerically) leaks into the cavity, and that the CFIE produces accurate solutions free from spurious oscillations.

2.3.6 The marching-on-in-time method for acoustic scattering analysis

To solve Eq. (2.9) numerically, \( \varphi'(r, t) \) on \( S \) is represented in terms of spatial and temporal basis functions \( f_n(r) \), \( n = 1, \ldots, N_s \), and \( T_i(t) \), \( i = 1, \ldots, N_t \), as

\[
\varphi'(r, t) = \sum_{i=1}^{N_t} \sum_{n=1}^{N_s} \varphi_{i,n} f_n(r) T_i(t),
\]

(2.13)

where \( \varphi_{i,n} \) are unknown expansion coefficients. Assuming that the spectral content of the incident field vanishes for \( \omega > \omega_{\text{max}} \), an accurate representation of \( \varphi'(r, t) \) can be obtained by choosing the number of basis functions as \( N_s \propto S_A(\omega_{\text{max}}/c)^2 \) and \( N_t \propto T \omega_{\text{max}} \), where \( S_A \) denotes the surface area of the scatterer and \( T \) the duration of the analysis. In this study, \( S \) is modeled as a collection of \( N_s \) flat triangular facets \( S_n \), \( n = 1, \ldots, N_s \), and the \( f_n(r) \) are chosen to be unit pulse functions defined as

\[
f_n(r) = \begin{cases} 
1 & ; r \in S_n \\
0 & ; \text{elsewhere}
\end{cases}
\]

(2.14)

The temporal basis functions are chosen to be cubic interpolation functions defined as \( T_i(t) = T(t - t_i) \), where \( t_i = i \Delta_t \) and

\[
T(t) = \begin{cases} 
1 + \frac{11}{6}(t/\Delta_t) + (t/\Delta_t)^2 + \frac{1}{6}(t/\Delta_t)^3 & ; -\Delta_t \leq t \leq 0 \\
1 + \frac{1}{2}(t/\Delta_t) - (t/\Delta_t)^2 - \frac{1}{2}(t/\Delta_t)^3 & ; 0 \leq t \leq \Delta_t \\
1 - \frac{11}{6}(t/\Delta_t) - (t/\Delta_t)^2 - \frac{1}{6}(t/\Delta_t)^3 & ; \Delta_t \leq t \leq 2\Delta_t \\
1 - \frac{1}{2}(t/\Delta_t) - (t/\Delta_t)^2 + \frac{1}{2}(t/\Delta_t)^3 & ; 2\Delta_t \leq t \leq 3\Delta_t \\
0 & ; \text{elsewhere}
\end{cases}
\]

(2.15)
The function $T(t)$ is an extension of the quadratic basis function proposed by Manara et al. [46]. Substituting Eq. (2.13) into Eq. (2.9) and testing the resulting equation at time $t = t_j$ with each of the $f_m(r), \ m = 1, \ldots, N_s$, the following matrix equation is obtained:

$$\bar{Z}_0 \Phi_j = F_j - \sum_{l=1}^{j-1} \bar{Z}_l \Phi_{j-l}, \quad (2.16)$$

where the elements of the vectors $\Phi_j$ and $F_j$ and the interaction matrix $\bar{Z}_l$ are given by

$$\Phi_{j,n} = \varphi_{j,n} \quad (2.17)$$

$$F_{j,m} = \int_S dr \ f_m(r) \left[ (1 - \alpha) \partial_t \varphi^i(r, t) + \alpha c \hat{n} \cdot \nabla \varphi^i(r, t) \right]_{t=t_j} \quad (2.18)$$

$$\bar{Z}_{l,mn} = \int_S dr \ f_m(r) \left[ L_c(f_n(r)T_{j-l}(t)) \right]_{t=t_j} \quad (2.19)$$

The vector elements in Eq. (2.18) are evaluated by using a one-point approximation to the surface integral as

$$F_{j,m} = (1 - \alpha) A_m \partial_t \varphi^i(r_m^c, t)_{t=t_j} + \alpha c A_m \hat{n} \cdot \nabla \varphi^i(r_m^c, t_j), \quad (2.20)$$

where $A_m$ and $r_m^c$ denote the area and the location of the center of the $m^{th}$ triangular facet, respectively. Numerical evaluation of the matrix elements appearing in Eq. (2.19) is detailed in Section 2.3.8 for the sake of providing the continuity of the present discussion.

Equation (2.16) relates the velocity potential expansion coefficients at the $j^{th}$ time step to the incident field and the coefficients that model the potential at prior time steps. Hence, all the expansion coefficients $\varphi_{j,n}$ can be evaluated by starting at the first time step ($j = 1$), forming the right-hand side, and recursively solving Eq. (2.16) for all time steps. This procedure is known as the MOT scheme [31].

In the past, most MOT implementations relied on explicit time stepping, which not only considerably limited the allowed time step size but also often resulted in late time instabilities.
As mentioned above, it has been demonstrated that use of higher-order quadrature rules and/or larger time step sizes yields implicit schemes that are stable for all practical purposes [43, 45]. In the present study, the surface integrals over triangular domains are evaluated as described in Section 2.3.8, and the time step size is chosen— independent of the mesh size— as \( \Delta t = 2\pi/(\beta \omega_{\text{max}}) \). For all the examples that will be presented, \( \beta = 10 \) yielded stable results.

### 2.3.7 Numerical results involving acoustic scattering

In order to study the quality of the solutions to the different time domain integral equations presented above, plane wave scattering from several rigid bodies is analyzed. The results obtained by solving Eq. (2.9), which reduces to Eq. (2.4) for \( \alpha = 0 \) and to Eq. (2.7) for \( \alpha = 1 \), are compared with solutions obtained either analytically or using a frequency domain boundary element method (FD-BEM) code, which has been validated independently. The FD-BEM employed uses the same spatial basis functions as the MOT scheme and circumvents the nonuniqueness problem by using the Burton-Miller approach. The wave speed of the medium is assumed to be \( c = 343 \) m/s. In all MOT analyses that will be presented, the matrix equation in Eq. (2.16) is solved using a transpose-free quasi-minimal residual algorithm [82]. The initial guess vector for this solver is set equal to the \( \Phi \) vector of the previous time step. With this initial guess, a relative residual error of \( 10^{-6} \) was obtained in less than 15 iterations for all cases presented below. In all MOT simulations, the excitation is a plane wave defined as

\[
\varphi^i(\mathbf{r}, t) = \cos\left[\omega_o\left(t - \mathbf{r} \cdot \hat{\mathbf{k}}^i/c\right)\right] \exp\left[-\left(t - \mathbf{r} \cdot \hat{\mathbf{k}}^i/c - 6\sigma\right)^2/2\sigma^2\right],
\]

(2.21)

where \( \omega_o \) is the pulse's center frequency, \( \sigma \) is a measure of the pulse's temporal width, and \( \hat{\mathbf{k}}^i \) is the propagation direction of the incident wave. This pulse can be assumed to be bandlimited to \( \omega_{\text{max}} = \omega_o + 6/\sigma \) since at that frequency the spectral content of the pulse is down by 78 dB from its peak value at \( \omega_o \).
Figure 2.2: (a) Magnitude of the total surface velocity potential $|\varphi^I(r,t)|$ at $\theta = 42.5^\circ$ for a rigid sphere modeled with 2210 triangular facets calculated using the Kirchhoff integral equation ($\alpha = 0$), its normal derivative ($\alpha = 1$), and the CFIE with $\alpha = 0.5$. (b) Magnitude of the Fourier transform of the differences between the $\alpha = 0.1$ solutions and the $\alpha = 0.5$ solution plotted in part (a).

As a first example, scattering from a sphere of radius 1 m centered at the origin and modeled by 2210 triangular facets is analyzed (see inset of Figure 2.2(a)). The incident pulse parameters are chosen as $\omega_o = 2\pi \times 200$ rad/s, $\sigma = 7.639$ ms, and $\mathbf{k}^i = -\hat{z}$ so that the two soft cavity modes at 171.5 Hz and 245.3 Hz and the rigid cavity modes at 182.4 Hz, 245.3 Hz, and 246.4 Hz could possibly be excited. The magnitude of the calculated velocity potential on the sphere at an arbitrary angle of $\theta = 42.5^\circ$ is shown in Figure 2.2(a) as a function of time for $\alpha = 0, 0.5,$ and 1. Clearly, the solutions for both $\alpha = 0$ and $\alpha = 1$ are corrupted by oscillations whereas the solution obtained with $\alpha = 0.5$ is free of any resonances. The spectra of the oscillations that corrupt the solutions to the $\alpha = 0$ and $\alpha = 1$ cases are obtained by Fourier transforming the difference of these solutions from the $\alpha = 0.5$ solution, and their magnitudes are plotted in Figure 2.2(b). Clearly the frequencies of oscillations that corrupt the solutions coincide with those of the cavity modes.
The effects of the internal resonances on the scattered fields are studied by extracting the scattering cross-section (SCS) of the scatterer at several frequencies from the time domain far fields computed via MOT and comparing them with analytical calculations using the Mie series. As seen in Figure 2.3, the SCS data extracted from the MOT analysis with $\alpha = 0.5$ shows perfect agreement with the analytical results, verifying that the potential all over the sphere has been calculated accurately for all time steps. Figure 2.3(c) shows that the SCS obtained by the normal derivative of the Kirchhoff integral equation ($\alpha = 1$) agrees well with the analytical values, although the surface potential values were corrupted with a cavity mode at 245.3 Hz. On the other hand, for $\alpha = 0$, the resonances at 171.5 and 245.3 Hz dramatically affect the calculated scattered fields (Figures 2.3(a) and (c)) and have almost no effect at other frequencies (Figure 2.3(b)). These results can be explained as follows. The resonant fields trapped inside the cavity have to be sustained by equivalent surface sources that reside on $S$, and the pressure field will be discontinuous across $S$ by an amount proportional to the strength of these sources. In the formulation of the Kirchhoff integral equation ($\alpha = 0$), $p^s(r, t) = -p^s(r, t)$ is enforced on $S_-$; hence, the same condition will not hold true on $S_+$. Therefore, there will be a field induced external to $S$ by the resonant component of the equivalent surface sources, and the scattered fields will also be corrupted. On the other hand, in the formulation of the normal derivative of the Kirchhoff integral equation, Eq. (2.6) enforces $\hat{n} \cdot \mathbf{v}^s(r, t) = -\hat{n} \cdot \mathbf{v}^s(r, t)$ on $S_+$ regardless of the discontinuity of the pressure field across $S$. Therefore, by the uniqueness of the solutions to the exterior problem, the corruption of the surface fields by resonant modes will not influence the scattered fields if Eq. (2.7) is used for the transient analysis. Although this final observation regarding the normal derivative of the Kirchhoff integral equation is justified by many numerical examples, there are cases when the corruption of the surface fields reflects itself on the radiated fields. However, in general, the scattered fields computed via the normal derivative of the Kirchhoff integral equation tend to be affected far less by the presence of resonant modes.

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Figure 2.3: Scattering cross-section of a unit sphere extracted from the MOT simulations compared against the analytical values at (a) 171.5 Hz, (b) 205 Hz, and (c) 245.3 Hz.
Analysis of scattering from a rigid cube with side length of 1 m constitutes the second example. The structure is modeled in terms of 2028 triangular facets, and it is positioned such that the cube's edges coincide with the coordinate axes as shown in the inset of Figure 2.4. The incident pulse is defined as \( \omega_o = 2\pi \times 350 \) rad/s, \( \sigma = 7.639 \) ms, and \( \hat{k}^i = -\hat{z} \). As can be verified from Figure 2.4, the surface fields oscillate at about 297 and 343 Hz for \( \alpha = 0 \) and \( \alpha = 1 \), respectively. Furthermore, for \( \alpha = 1 \), the oscillations, which are distinctly different from the late time instabilities of the explicit MOT schemes, seem to grow with a slowly varying envelope. However, no spurious oscillations are present in the solution obtained by setting \( \alpha = 0.5 \). The \( xz \)-plane SCS data extracted from the MOT simulation are compared to the SCS obtained by the FD-BEM solver. As seen in Figure 2.5, the agreement between the \( \alpha = 0.5 \) and FD-BEM results is excellent at both 297 and 343 Hz. As expected, the solution with \( \alpha = 0 \) deviates considerably from the others at 297 Hz while the corruption of the surface fields has some effect on the radiated fields for \( \alpha = 1 \) at 343 Hz.

![Magnitude of the total surface velocity potential](image)

**Figure 2.4:** Magnitude of the total surface velocity potential \( |\varphi'(r,t)| \) on the surface of a rigid cube modeled with 2028 triangular facets calculated using the Kirchhoff integral equation \( (\alpha = 0) \), its normal derivative \( (\alpha = 1) \), and the CFIE with \( \alpha = 0.5 \).
Figure 2.5: Scattering cross-section of a unit cube in the xz-plane extracted from the MOT simulations compared against those obtained by the FD-BEM at (a) 297 Hz and (b) 343 Hz.

As the last example, scattering from an almond modeled with 2572 triangular facets is studied. The scatterer, which is depicted in the inset of Figure 2.6(a), fits into a box of dimensions $3 \times 1.15 \times 0.4$ m and is a modified version of the almond described in [83]. The parameters characterizing the incident pulse are $\omega_0 = 2\pi \times 500$ rad/s, $\sigma = 7.639$ ms, and $\hat{k} = -\hat{z}$. As seen in Figure 2.6(a), spurious oscillations not only corrupt the surface fields for $\alpha = 0$ and $\alpha = 1$, but they also show an exponential increase in amplitude. Thorough experimentation has shown that this type of instability, unlike the high-frequency instabilities encountered with explicit MOT schemes, cannot be avoided by increasing the time step size. Furthermore, if only the Helmholtz integral equation or its derivative is used, the system matrix of the FD-BEM becomes ill-conditioned at about $505$ Hz, which corresponds to the frequency of the oscillations in the MOT solutions. Thus, it can be inferred that internal cavity modes exist in the vicinity of this frequency. As seen in Figure 2.6(a), setting $\alpha = 0.5$ yields a stable solution that is also free of sustained oscillations. In Figure 2.6(b), the SCS of the scatterer in the xz-plane extracted from the time domain simulation results at $505$ Hz is again compared to that
computed via the FD-BEM that employed the Burton-Miller approach. The perfect match between the $\alpha = 0.5$ case and the FD-BEM result, together with the fact that the $\alpha = 0$ case deviates considerably more than the $\alpha = 1$ case, also supports the conclusion that the observed instabilities were due to the presence of resonant modes that were avoided by setting $\alpha = 0.5$. Finally, although all the resonance-free results presented here were obtained by setting $\alpha = 0.5$, tests with other values of $\alpha$ in the range $0 < \alpha < 1$ were conducted and, as expected, the results were observed to be stable and devoid of sustained oscillations except for $\alpha < 0.1$ and $\alpha > 0.9$.

Figure 2.6: (a) Magnitude of the total surface velocity potential $|\phi'(r,t)|$ on the surface of a rigid almond modeled with 2572 triangular facets using different $\alpha$ values. (b) The SCS data in the $xz$-plane obtained from the MOT simulations and the FD-BEM at 505 Hz.

2.3.8 Numerical evaluation of interaction matrix elements

In this section, the numerical evaluation of the vector and matrix elements given in Eqs. (2.18) and (2.19) is elucidated.

The matrix elements $\mathbf{Z}_{l,mn}$ are evaluated as follows. Using Eqs. (2.4), (2.7), and (2.9) in Eq. (2.19) yields
\[ \mathcal{Z}_{l,m} = -(1 - \alpha) \int_{S_m} dr \mathcal{L}_p \{ f_n(\mathbf{r}) T_{j-l}(t) \} \bigg|_{t=t_j} + \alpha c \int_{S_m} dr \mathcal{L}_v \{ f_n(\mathbf{r}) T_{j-l}(t) \} \bigg|_{t=t_j}, \]  

(2.22)

where

\[ \int_{S_m} dr \mathcal{L}_p \{ f_n(\mathbf{r}) T_{j-l}(t) \} \bigg|_{t=t_j} = \begin{cases} \frac{A_m}{2} \partial_t T_{j-l}(t) \bigg|_{t=t_j} & ; \quad m = n \\ \int_{S_m} \int_{S_n} dr \partial_t T_{j-l}(t - R/c) \bigg|_{t=t_j} & ; \quad m = n \end{cases} \]  

(2.23)

\[ \int_{S_m} dr \mathcal{L}_v \{ f_n(\mathbf{r}) T_{j-l}(t) \} \bigg|_{t=t_j} = -\int_{S_m} \int_{S_n} dr \partial_t \mathbf{\hat{n}} \cdot \nabla \mathbf{\hat{n}}' \cdot \nabla' \frac{T_{j-l}(t - R/c)}{4\pi R} \bigg|_{t=t_j}. \]  

(2.24)

The integrands in Eqs. (2.23) and (2.24) can also be written as

\[ \mathbf{\hat{n}}' \cdot \nabla' \frac{\partial_t T_{j-l}(\tau)}{4\pi R} = \mathbf{\hat{n}}' \cdot \mathbf{\hat{R}} \left[ \frac{\partial^2 T_{j-l}(\tau)}{c R} + \frac{\partial\tau T_{j-l}(\tau)}{R^2} \right] \]  

(2.25)

\[ \mathbf{\hat{n}} \cdot \nabla \mathbf{\hat{n}}' \cdot \nabla' \frac{T_{j-l}(\tau)}{4\pi R} = -\left( \mathbf{\hat{n}} \cdot \mathbf{\hat{R}} \right) \left( \mathbf{\hat{n}}' \cdot \mathbf{\hat{R}} \right) \left[ \frac{\partial^2 T_{j-l}(\tau)}{c^2 R} + \frac{3 \partial\tau T_{j-l}(\tau)}{c R^2} + \frac{3 T_{j-l}(\tau)}{R^3} \right] \]  

(2.26)

In the above equations, \( \tau = t - R/c \), \( \mathbf{\hat{R}} = (\mathbf{r} - \mathbf{r}')/R \), and analytical expressions for each term with time derivatives can be readily obtained via Eq. (2.15). When \( m = n \) in Eq. (2.24), the integrand becomes hypersingular. By approximating the outer integral with a one-point rule, using Eq. (2.26), and applying a limiting procedure similar to that performed by Terai [84] in the frequency domain, the finite part of this hypersingular integral can be evaluated as

\[ \int_{S_n} dr \mathcal{L}_v \{ f_n(\mathbf{r}) T_{j-l}(t) \} \bigg|_{t=t_j} = A_n \int_0^{2\pi} d\theta \frac{T_{j-l}(t_j - \rho_n(\theta)/c)}{\rho_n(\theta)} + \frac{2\pi}{c} \partial_t T_{j-l}(t) \bigg|_{t=t_j}, \]  

(2.27)

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where \( \theta \) and \( \rho_n(\theta) \) are depicted in Figure 2.7. The contour integral in the first term can be evaluated analytically, although using a seventeen-point Gauss-Legendre quadrature rule on each edge of the triangular facet yields values of comparable accuracy.

![Figure 2.7: Definition of the variables \( \theta \) and \( \rho_n(\theta) \) used in the evaluation of the contour integral in Eq. (2.27).](image)

The surface integrals over triangular domains which appear in Eqs. (2.23) and (2.24) for the nonself terms \((m \neq n)\) are evaluated numerically. Different quadrature rules are used for adjacent and separated facet pairs. If the source and observation domains are not adjacent, i.e., if \( S_m \) and \( S_n \) do not have a common vertex, the integration over \( S_m \) is approximated with a one-point rule and the integral over \( S_n \) is evaluated using a seven-point Gaussian quadrature rule (see Section 25.4.63 in [85]). For the adjacent facet pairs, different rules are used for evaluating Eqs. (2.23) and (2.24) accurately. In Eq. (2.24), the \( S_m \) integration is still approximated using a one-point rule. However, the integral over the source facet is evaluated by first subdividing \( S_n \) into four triangles by connecting the centers of the edges and then applying the seven-point quadrature rule on each of these four triangles. In Eq. (2.23), the seven-point rule is used for evaluating both integrals. While applying all the aforementioned quadrature rules, the retarded time between each source-observer quadrature point pair is calculated separately.
2.4 Time Domain Integral Equations for Electromagnetics

In this section, the problems stemming from the presence of resonant modes in the transient analysis of electromagnetic wave scattering from three-dimensional perfect electrically conducting (PEC) bodies described by closed surfaces are studied, and a time domain CFIE is proposed as a remedy. Section 2.4.1 introduces the problem to be solved. Derivations of the time domain EFIE and MFIE that have previously been used to solve this problem are reviewed in Sections 2.4.2 and 2.4.3, respectively. The time domain CFIE is introduced in Section 2.4.4. Energy arguments are invoked in Section 2.4.5 to determine whether these integral equations can sustain internal fields. Section 2.4.6 outlines the formulation of the MOT method used in this study to solve the aforementioned integral equations. Numerical results demonstrating the effectiveness of the CFIE in improving the accuracy of the solutions are presented in Section 2.4.7.

2.4.1 Description of the electromagnetic scattering problem

Let $S$ denote the surface of a closed perfectly conducting body that resides in free space and that is excited by a transient electromagnetic field $\{E^i(r,t), H^i(r,t)\}$ (Figure 2.8). The interaction of the incident field with $S$ results in a surface current $J(r,t)$, which in turn generates a scattered electromagnetic field $\{E^s(r,t), H^s(r,t)\}$. It is assumed that the incident field does not interact with $S$ prior to time $t = 0$ and therefore $J(r,t) = 0$ for $t < 0$. The scattered fields are fully characterized by the vector and scalar potentials defined as

$$A(r, t) = \frac{\mu_0}{4\pi} \int_S dr' \frac{J(r', t)}{R}$$

and

$$A(r, t) = \frac{\mu_0}{4\pi} \int_S dr' \frac{J(r', t)}{R}$$

(2.28)
\[
\phi(r, t) = \frac{1}{4\pi \varepsilon_0} \int_\mathcal{S} \frac{\rho(r', \tau)}{R} \, d\tau',
\]

(2.29)

where \( R = |\mathbf{R}| = |\mathbf{r} - \mathbf{r}'| \) is the distance between the source and observation points, \( \tau = t - R/c \) denotes retarded time, \( c \) is the speed of light, and \( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and permeability of free-space, respectively. Also, \( \rho(r, t) \) denotes the surface charge density which is related to \( \mathbf{J}(r, t) \) through the continuity equation

\[
\nabla \cdot \mathbf{J}(r, t) + \partial_t \rho(r, t) = 0.
\]

(2.30)

In what follows, surfaces conformal to \( \mathcal{S} \), but residing just outside and just inside \( \mathcal{S} \), are denoted by \( \mathcal{S}_+ \) and \( \mathcal{S}_- \), respectively, and \( \hat{\mathbf{n}} \) denotes an outward pointing and position-dependent unit normal to \( \mathcal{S} \). Next, three time domain integral equations that relate the surface current \( \mathbf{J}(r, t) \) to the incident field \( \{\mathbf{E}^i(r, t), \mathbf{H}^i(r, t)\} \) will be derived by imposing appropriate boundary conditions on \( \mathcal{S}, \mathcal{S}_- \) or \( \mathcal{S}_+ \).

![Figure 2.8: Description of the electromagnetic surface scattering problem.](image)

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2.4.2 Time domain electric field integral equation

To arrive at a time domain EFIE, note that the scattered electric field \( E^s(r,t) \) can be expressed in terms of the above potentials as

\[
E^s(r,t) = -\partial_t A(r,t) - \nabla \phi(r,t).
\]  
(2.31)

The incident field \( E^i(r,t) \) can be related to the scattered field by enforcing the boundary condition that the total electric field \( E^i(r,t) = E^i(r,t) + E^s(r,t) \) tangential to \( S \) vanishes:

\[
\hat{n} \times \hat{n} \times E^i(r,t) = -\hat{n} \times \hat{n} \times E^s(r,t) \quad r \in S, S_-, S_+.
\]  
(2.32)

Note that the same condition holds true on both \( S_- \) and \( S_+ \) due to the continuity of the tangential electric field. Hence, using Eqs. (2.28)-(2.31) in Eq. (2.32), the following EFIE for \( J(r,t) \) is obtained:

\[
\hat{n} \times \hat{n} \times E^i(r,t) = \hat{n} \times \hat{n} \times \left\{ \frac{\mu_o \partial_t}{4\pi} \oint_{S'} \frac{J(r',\tau)}{R} \, dS' - \frac{\nabla}{4\pi \varepsilon_o} \oint_{S'} \frac{\nabla' \cdot J(r',t')}{R} \, dt' \right\} \quad r \in S, S_-, S_+.
\]  
(2.33)

The RTBI operator \( L_e \) defined in the above equation will be further referred to as the electric field operator. Note that, assuming that no initial conditions on \( J(r,t) \) are imposed, there exist many \( J(r,t) \neq 0 \) that satisfy the homogenous equation \( L_e \{ J(r,t) \} = 0 \). These fields are superpositions of the resonant modes of a PEC cavity bounded by the surface \( S \) [74]. However, these solutions to Eq. (2.33) are ruled out since they do not satisfy the condition that \( J(r,t) = 0 \) on \( S \) for \( t < 0 \).
2.4.3 Time domain magnetic field integral equation

Likewise, a time domain MFIE can be derived by expressing the scattered magnetic field as

$$\mathbf{H}^s(r, t) = \frac{1}{\mu_0} \nabla \times \mathbf{A}(r, t)$$  \hspace{1cm} (2.34)

and by enforcing the condition that the total magnetic field $\mathbf{H}^t(r, t) = \mathbf{H}^s(r, t) + \mathbf{H}^f(r, t)$ tangential to $S_-$ vanishes, i.e.,

$$\hat{n} \times \mathbf{H}^f(r, t) = -\hat{n} \times \mathbf{H}^s(r, t) \quad r \in S_-.$$  \hspace{1cm} (2.35)

Upon using Eq. (2.28) together with Eq. (2.34) in Eq. (2.35), the following MFIE for $\mathbf{J}(r, t)$ results:

$$\hat{n} \times \mathbf{H}^t(r, t) = -\frac{1}{4\pi} \hat{n} \times \oint_S \left[ \frac{\partial}{\partial t} \mathbf{J}(r', \tau) + \frac{1}{c^2} \frac{1}{R^2} \mathbf{J}(r', \tau) \right] \times \hat{R} \quad r \in S_-,$$

$$\hspace{3cm} \approx \mathcal{L}_h\{\mathbf{J}(r, t)\}$$  \hspace{1cm} (2.36)

where $\hat{R} = \mathbf{R}/R$. Henceforth, the RTBI operator $\mathcal{L}_h$ defined in the above equation will be referred to as the magnetic field operator. Again, there exist many $\mathbf{J}(r, t) \neq 0$ that satisfy the homogenous equation $\mathcal{L}_h\{\mathbf{J}(r, t)\} = 0$; these fields are the superpositions of the resonant modes of a perfect magnetically conducting (PMC) cavity bounded by $S$ [74]. However, again, these solutions to Eq. (2.36) do not satisfy the initial conditions and therefore should be ruled out.

2.4.4 Time domain combined field integral equation

It has been pointed out in Section 2.2 that the solutions to the homogeneous time domain EFIE and MFIE can be corrupted by (numerical) excitation of the resonant cavity modes characterized by the poles of the resolvent of $\mathcal{L}_e$ and $\mathcal{L}_h$ [77], respectively. While this does not
necessarily lead to unstable behavior when implicit methods are used, it does follow that the solution can be corrupted by the presence of perturbed cavity modes that can be supported by EFIE or MFIE. It is well known that the solutions to both the frequency domain EFIE and MFIE can be corrupted when the frequency of the incident field approaches that of one of the scatterer's resonant modes. It is also known that this interior resonance problem in the frequency domain can be remedied using a variety of approaches. Chief among those, and perhaps most widely accepted, is the use of a frequency domain CFIE [74, 75]. Next, the time domain counterpart of the CFIE is derived and its resonance suppressing property is discussed.

The time domain CFIE is constructed by combining the EFIE and MFIE as

$$\left(-\frac{\beta}{\eta_o}\right) \hat{n} \times \mathbf{E}^i(r, t) + \hat{n} \times \mathbf{H}^i(r, t) = 0 \quad r \in S_-$$

(2.37)

where $\eta_o = \sqrt{\mu_o/\varepsilon_o}$ is the intrinsic impedance of the free space, introduced in Eq. (2.37) for scaling purposes and $\beta$ is a (real) constant that is greater than zero. In terms of incident and scattered fields, the above equation can be stated as

$$\left(-\frac{\beta}{\eta_o}\right) \hat{n} \times \mathbf{E}^i(r, t) + \hat{n} \times \mathbf{H}^i(r, t) = \left(-\frac{\beta}{\eta_o}\right) \mathcal{L}_e\{\mathbf{J}(r, t)\} + \mathcal{L}_h\{\mathbf{J}(r, t)\}$$

$$= \mathcal{L}_c\{\mathbf{J}(r, t)\} \quad r \in S_-.$$ 

(2.38)

Note that, the combined field RTBI operator $\mathcal{L}_c$ is a linear combination of the electric and magnetic field operators $\mathcal{L}_e$ and $\mathcal{L}_h$, and that Eq. (2.38) reduces to Eq. (2.36) for $\beta = 0$ and to Eq. (2.33) as $\beta \to \infty$.

2.4.5 Corruption of numerical solutions to time domain integral equations by internal fields

As pointed out before, when numerically solving Eqs. (2.33) and (2.36), poles corresponding to the resonance frequencies of the cavity formed by $S$ can be excited, leading to
the generation of currents that produce energy inside $S$. To examine the effect of numerical leakage of fields into $S$ on the solution to either the EFIE or MFIE, assume that at time $t = t_0$ a field exists inside the cavity. Do these equations provide an \textit{intrinsic} mechanism for expelling this energy out of $S$? From the law of conservation of energy it follows that

$$W(t) = W(t_0) - \int_{t_0}^{t} \int \overbrace{\mathbf{n} \cdot \left( \mathbf{E}^i(r, t') \times \mathbf{H}^i(r, t') \right)}_{S_-} dt' \, d\mathbf{r}$$

(2.39)

where

$$W(t) = \frac{1}{2} \int_{V_-} \left( \varepsilon_0 \left| \mathbf{E}^i(r, t) \right|^2 + \mu_0 \left| \mathbf{H}^i(r, t) \right|^2 \right)$$

(2.40)

is the energy stored inside the volume $V_-$ enclosed by $S_-$ at time $t$. Next, the EFIE, MFIE and CFIE equations are compared based upon their treatment of fields interior to $S$. Equation (2.39) can be rewritten as

$$W(t) = W(t_0) - \int_{t_0}^{t} \int \overbrace{\mathbf{n} \cdot \mathbf{H}^i(r, t') \cdot \left( \mathbf{E}^i(r, t') \times \mathbf{n} \right)}_{S_-} dt' \, d\mathbf{r}$$

(2.41)

$$= W(t_0) - \int_{t_0}^{t} \int \overbrace{\mathbf{E}^i(r, t') \cdot \left( \mathbf{H}^i(r, t') \times \mathbf{n} \right)}_{S_-} dt' \, d\mathbf{r}$$

As either $\mathbf{n} \times \mathbf{E}^i(r, \tau) = 0$ or $\mathbf{n} \times \mathbf{H}^i(r, \tau) = 0$ is imposed on $S_-$ in deriving the EFIE or MFIE, it can be deduced that the solutions to both these equations will be such that $W(t) = W(t_0) > 0$. In other words, both these equations do not provide an intrinsic mechanism for reducing energy stored in the cavity. \textit{Hence, numerical leakage of energy} into (and out of) the cavity is at the mercy of the implementation details of the computational solution scheme. On the other hand, the time domain CFIE does provide a mechanism for expelling energy out of $S$, as the time
domain CFIE does not support resonant modes. Using Eq. (2.37) together with Eq. (2.41) it can be shown that

\[ W(t) = W(t_0) - \frac{\beta}{\eta_o} \int_{t_0}^{t} \int_{S_\omega} \left| \hat{\mathbf{n}} \times \mathbf{E}^f(\mathbf{r}, t') \right|^2 dt' \right) dr. \tag{2.42} \]

Now, if \( W(t_0) > 0 \), then \( \hat{\mathbf{n}} \times \mathbf{E}^f(\mathbf{r}, t) = -\left( \eta_o / \beta \right) \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H}^f(\mathbf{r}, t) \) cannot be zero for all \( t > t_0 \), because any field residing inside a closed body will interact with the walls at some future time. At this time, by virtue of Eq. (2.42), the energy inside the cavity will be reduced. This process will continue until \( W(t_0) = 0 \).

Of course, the above arguments provide no more than a heuristic picture the CFIE behavior. This section is, like many other publications on this topic, largely experimental in nature. The above argument implies solely that the time domain CFIE provides an intrinsic mechanism for expelling the energy out of the cavity (very much like frequency domain CFIEs). Our numerical results indicate that, for all structures tested and in contrast to the EFIE and MFIE, this loss mechanism outweighs numerical leakage of energy into the cavity and thereby suppresses any energy buildup inside the closed body.

An earlier investigation into the use of the CFIE was presented by Vechinski and Rao in [37]. There, the authors used

\[ -(\beta / \eta_o) \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \partial_t \mathbf{E}^f(\mathbf{r}, t) + \hat{\mathbf{n}} \times \partial_t \mathbf{H}^f(\mathbf{r}, t) = \partial_t \mathcal{L}_c \{ \mathbf{J}(\mathbf{r}, t) \} \quad \mathbf{r} \in S_\omega \tag{2.43} \]

for the analysis of 2-D scattering instead of Eq. (2.38). The purpose of the analysis presented in [37] was to determine whether the late-time instabilities commonly encountered in MOT schemes that rely on explicit time stepping can be overcome by using a combined field formulation. In this regard, no benefit from using a CFIE was observed. This can again be explained using the arguments put forth by Rynne and Smith [34]. A CFIE formulation only

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eliminates the poles that lie on the imaginary axis corresponding to the interior cavity modes, and does not affect the location of the poles describing the exterior problem. As the MOT scheme prescribed in [37] relies on an explicit time-marching scheme, it is conjectured that the poles of the resolvent of $\partial_t \mathcal{L}_e$ that lie close to the imaginary axis easily shift into the right half-plane, thereby generating instabilities. As a result, little benefit is observed from using a resonance suppressing integral equation method. Finally, experiments indicate that our implicit time stepping scheme for solving Eq. (2.38) yields a more accurate solution than that of Eq. (2.43), based upon otherwise very similar implementation choices; it is for this reason that in this section Eq. (2.38) was selected over Eq. (2.43).

2.4.6 The marching-on-in-time method for electromagnetic scattering analysis

Equation (2.38) can be solved numerically by adopting a discrete representation for the current $\mathbf{J}(r,t)$ and by discretizing the CFIE accordingly. Following standard practice, the scatterer is modeled by flat triangular facets and the current $\mathbf{J}(r,t)$ is represented using spatial and temporal basis functions $f_n(r)$ for $n = 1, \ldots, N_s$ and $T_j(t)$ for $j = 1, \ldots, N_t$, such that

$$\mathbf{J}(r,t) = \sum_{j=0}^{N_t} \sum_{n=1}^{N_s} I_{n,j} f_n(r) T_j(t),$$

(2.44)

where $I_{n,j}$ is the weight associated with the space-time basis function $f_n(r)T_j(t)$. In our implementation, the $f_n(r)$ are chosen to be the Rao-Wilton-Glisson (RWG) basis functions, which have been used extensively in the integral equation-based analysis of both time-harmonic and transient scattering phenomena [4, 86]. Each of these basis functions is associated with each edge joining two triangles, and is defined as
$$f_n(r) = \begin{cases} \frac{l_n}{2A_n^+} \rho_n^+(r) & ; \quad r \in \Gamma_n^+ \\ \frac{l_n}{2A_n^-} \rho_n^-(r) & ; \quad r \in \Gamma_n^- \\ 0 & ; \quad \text{elsewhere}, \end{cases}$$

(2.45)

where $l_n$ is the length of the common edge between the triangles $\Gamma_n^+$ and $\Gamma_n^-$, $A_n^\pm$ is the area of the triangle $\Gamma_n^\pm$, and $\rho_n^+(r)$ is the position vector with respect to the free vertex of the corresponding triangle. The temporal bases $T_j(t) = T(t - j\Delta_t)$ for a time step size $\Delta_t$ are assumed to be triangular functions [4]. In other words, $T_j(t) = 1$ for $t = 0$, linearly interpolates to zero at $t = \pm\Delta_t$, and is zero outside the interval $(-\Delta_t, \Delta_t)$. It is possible to choose higher-order spatial and temporal interpolation functions to improve the accuracy and stability of the MOT scheme [41, 46]. By expanding the current density as in Eq. (2.44), and by applying a spatial Galerkin testing procedure at $t = t_j = j\Delta_t$, the following matrix equation is obtained:

$$\overline{Z}_0 I_j = F_j^{inc} - \sum_{l=1}^{j} \overline{Z}_l I_{l-j} \quad (2.46)$$

where the entries of the vectors $I_j$, $F_j^{inc}$ and the matrices $\overline{Z}_l$ are given by

$$I_{j,m} = I_{m,j}, \quad (2.47)$$

$$F_{j,m}^{inc} = -\left(\beta / \eta_o\right) \left( \langle f_m(r), \hat{n} \times \hat{n} \times E_i(r, t_j) \rangle + \langle f_m(r), \hat{n} \times H_i(r, t_j) \rangle \right), \quad (2.48)$$

$$\overline{Z}_{l,mn} = \left( \langle f_m(r), L_c \{ j_n(r) T_{j-l}(t) \} \rangle \right)_{t=t_j}. \quad (2.49)$$

In the above equations, $\langle \Phi(r), \Psi(r) \rangle = \int_S dr \Phi(r) \cdot \Psi(r)$ denotes the standard inner product. Equation (2.46) is the basis for the classical MOT scheme. Assuming that the currents up to the $(j - 1)^{th}$ time step are known, this equation permits the computation of the currents associated

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with the \( j^{th} \) time step. Hence, the currents at all time points of interest can be computed recursively. In our practical implementation of this scheme, implicit time stepping, and accurate spatial and temporal integration rules are used to ensure stability of the MOT algorithm. More specifically, all inner products are evaluated by sampling currents and fields at the centroids of the triangular facets, and using a one-point integration rule [86]. This testing procedure leads to an implicit scheme, even for the time step size suggested in [4]. In our implementation, the time step size chosen is independent of the spatial discretization and is \( \Delta t = 1/(\beta f_{\text{max}}) \), where \( \beta \geq 10 \). The resulting scheme is termed implicit because \( Z_0 \) is not diagonal. However, as this matrix is highly sparse, a nonstationary iterative solver such as QMR (Quasi Minimal Residual) [87] can be used to efficiently solve for \( I_j \) each and every time step. The additional cost incurred from the use of such a solver is insignificant when compared to the overall cost. In addition to implicit time stepping, accurate spatial and temporal integration rules are used for computing \( L_c \{ J(r,t) \} \) [41, 85].

It has been observed that the MOT scheme for solving the MFIE is easier to stabilize than that for the EFIE. The studies of Manara et al. [46] and Rao and Sarkar [88] notwithstanding, we have found that implicit schemes and accurate integration rules do not guarantee stability when the spectrum of the incident pulse contains the resonance frequencies of the scatterer. To combat this EFIE instability, averaging as suggested by Rynne and Smith [34] is used, though it should be stressed, only for the EFIE. On the other hand, MOT schemes for solving the CFIE and MFIE have been found to be always stable without resorting to averaging schemes as long as implicit time stepping methods and accurate integration rules are used.
2.4.7 Numerical results involving electromagnetic scattering

The objective of this section is to demonstrate that the above proposed time domain CFIE yields solutions not corrupted by the presence of cavity modes even when the spectrum of the incident pulse includes one or more of the scatterer’s resonance frequencies. As a first step towards validating our codes, the current on an electrically small structure obtained using the CFIE code will be compared against that obtained using codes based upon the EFIE and MFIE presented in Eqs. (2.33) and (2.36), respectively. In this experiment, it will be ensured that no resonant modes are excited by virtue of the choice of dimensions of the scatterer with respect to the wavelength at the highest frequency present in the incident pulse (delineated by the notion of bandwidth to be introduced shortly). Next, the currents on, and the far-field signatures of, electrically large structures will be computed using the time domain CFIE code. The far-field signatures will then be Fourier transformed into the frequency domain and the object’s radar cross-section (RCS) will be extracted at several frequencies. These results will then be compared against the RCS computed using FISC (Fast Illinois Solver Code), a frequency domain fast multipole-based CFIE code that has been extensively validated [53]. The incident pulse used in all the examples that follow is a modulated Gaussian plane wave given by

\[ E^i(r,t) = \hat{p} \cos(2\pi f_o t) \exp \left[ -\frac{(t - t_p - r \cdot \hat{k}/c)^2}{2\sigma^2} \right], \tag{2.50} \]

where \( f_o \) is the center frequency, \( \hat{k} \) denotes the direction of travel of the incident wave and \( \hat{p} \) its polarization, \( \sigma = 6/(2\pi f_{bw}) \), \( t_p = 3.5\sigma \), and \( f_{bw} \) will be further referred to as the bandwidth of the signal. It is to be noted that the power in the incident pulse is down by 160 dB at \( f = f_o \pm f_{bw} \) relative to that at \( f_o \).
To validate the time domain CFIE against the time domain MFIE and EFIE, consider a cube of dimensions $1 \times 1 \times 1$ m shown in the inset in Figure 2.9. A modulated Gaussian plane wave with $f_o = 100$ MHz and $f_{bw} = 40$ MHz, traveling along $\mathbf{k} = -\mathbf{z}$ with $\mathbf{p} = \mathbf{i}$ is incident on the cube. It is ensured that $f = f_o + f_{bw}$ is less than 150 MHz, which is the frequency of the first resonant mode of the cube. In Figure 2.9 the magnitude of the current at a point on the cube's upper surface, computed using $\beta = 0.0$ (MFIE), 0.25, 1.0, 4.0, and $\infty$ (EFIE), is plotted against time. It is seen that the temporal signatures of the current computed using all values of $\beta$ agree well with each other. However, unlike the currents computed using the CFIE and MFIE, whose magnitude keeps decreasing with time, those obtained using the EFIE stabilize at a value three orders of magnitude below the peak. This behavior of the EFIE has also been observed by other researchers [42].

![Figure 2.9: Comparison of the currents observed on a cube as a function of time. The currents are computed using the time domain CFIE code for $\beta = 0.0$ (MFIE), 0.25, 1.0, 4.0, and $\infty$ (EFIE).](image)

Having ascertained that the numerical implementation of the time domain CFIE does yield a solution that coincides with that of the MFIE and the EFIE when no resonant modes are
excited, we next examine the CFIE’s performance when the spectrum of the incident pulse encompasses the body’s resonance frequencies. To this end, consider a sphere of radius \( r = 1 \) m, illuminated by a \( \hat{p} = \hat{x} \) polarized modulated Gaussian pulse with \( \hat{k} = -\hat{z} \). The pulse has a bandwidth of \( f_{bw} = 150 \) MHz, and a center frequency of \( f_0 = 200 \) MHz. The above choice of center frequency and bandwidth is such that the \( TM_{m,2,1} \) (184 MHz), \( TE_{m,1,1} \) (214 MHz) and \( TM_{m,3,1} \) (237 MHz) modes are excited. While the sphere theoretically also resonates at \( f = 131, 275, 289, 292, 333, 340 \) MHz, etc., these modes are barely excited as the power in the incident pulse at these frequencies is down by at least 30 dB from its peak at \( f_0 \). The magnitudes of the current at a point on the sphere with \( (\theta = 65^\circ, \phi = 72^\circ) \) obtained using the EFIE, MFIE, and CFIE (\( \beta = 0.25 \)) codes are compared in Figure 2.10(a). It is apparent that the solution to the MFIE exhibits a characteristic ringing, whereas that of the CFIE dies down. As mentioned earlier, since the spectrum of the incident pulse contains resonance frequencies characteristic of the body, it was necessary to use four-point temporal averaging [34] in addition to implicit time stepping to stabilize the solution to the EFIE. As a consequence of using the averaging process, which suppresses resonance effects to some extent, the EFIE solution does not exhibit the same features as that of the MFIE. However, both are still quite different from that obtained using the CFIE. The differences between the currents are further highlighted by examining their Fourier transforms. Figure 2.10(b) shows that the Fourier transform of the current obtained using the MFIE differs from that obtained using the CFIE, especially in the vicinity of the above mentioned resonance frequencies. Also, the Fourier transform of the current obtained using the EFIE is slightly smaller than that obtained using the CFIE and is distorted at the ends of the spectrum, both of which are consequences of temporal averaging. Examination of Figure 2.10(c), where the difference between the Fourier transforms of the current obtained using the MFIE and CFIE are plotted, reveals the presence of the principal resonant modes more clearly.
Next, the RCS patterns obtained using the MFIE and CFIE codes are compared against those obtained using FISC. The RCS patterns obtained using the EFIE codes are not shown as it is well known that, while the currents on the surface computed using the EFIE are corrupted by interior modes, the scattered far-fields obtained from them are not [74]. Figure 2.11 compares
Figure 2.11: The RCS of a sphere in the $xz$-plane extracted from the time domain CFIE and MFIE is compared to that obtained from FISC for four different frequencies. The incident wave propagates along $\hat{k} = -\hat{z}$ and is $\hat{p} = \hat{x}$ polarized.
the RCS patterns in the \(xz\)-plane computed using the time domain CFIE and MFIE codes and FISC at four different frequencies chosen either towards the end of the spectrum or close to a resonance. As is seen in these figures, the time domain CFIE faithfully reproduces the results obtained from FISC while the MFIE does not. It should be noted that the CFIE results agree reasonably well with those from FISC at both 120 MHz and 280 MHz in spite of the fact that at these frequencies the power in the incident field is down by 45 dB from its peak value. These results are not surprising, as the existence of nonphysical resonant currents in the solution to the MFIE will cause errors to propagate in any MOT scheme. Thus, when the RCS pattern is extracted from the far field signature, these errors are most conspicuous at the ends of the band. While comparison of MFIE, CFIE, and FISC results at other points in the range \(140 \text{ MHz} \leq f \leq 270 \text{ MHz}\) are omitted for the sake of brevity, they agree very well with each other as long as \(f\) is not close to any of the resonance frequencies.

Next, scattering from a cone-sphere is studied. The cone is 1 m long, the radius of the half sphere attached to the cone is 0.235 m, and the cone-sphere is discretized with 1656 unknowns. The incident field is a modulated Gaussian pulse with center frequency \(f_o = 400 \text{ MHz}\) and bandwidth \(f_{bw} = 350 \text{ MHz}\); it is \(\hat{p} = \hat{z}\) polarized, and is traveling in the \(\hat{k} = -\hat{x}\) direction. The RCS patterns in the \(xz\)-plane obtained from the time domain MFIE and CFIE codes are compared against those obtained from FISC for four different frequencies, as shown in Figure 2.12. As before, while the results obtained from the CFIE code (\(\beta = 1.0\)) agree very well with those from FISC for all four frequencies, those obtained from the MFIE code do not.
Figure 2.12: The RCS of a cone-sphere in the xz-plane, extracted from the time domain CFIE and MFIE is compared to that obtained from FISC for a set of frequencies. The incident wave is $\hat{p} = \hat{z}$ polarized and is traveling in the $\hat{k} = -\hat{x}$ direction.

In the next two examples the scatterer being analyzed is an almond with a maximum height of 0.0575 m, maximum width of 1.15 m, and length of 3 m. In the first of these two examples the incident wave propagates along $\hat{k} = -\hat{z}$, is $\hat{p} = \hat{y}$ polarized, has a center frequency of $f_o = 200$ MHz, and bandwidth of $f_{bw} = 150$ MHz. The RCS in the yz-plane is computed and representative results are shown in Figure 2.13. The agreement between the results obtained
Figure 2.13: The RCS of an almond in the xz-plane, extracted from the time domain CFIE and MFIE is compared to that obtained from FISC for a set of frequencies. The incident wave is \( \hat{p} = \hat{y} \) polarized and is traveling in the \( \hat{k} = -\hat{z} \) direction.

from the time domain CFIE code (\( \beta = 1.0 \)) and FISC is excellent for a broad range of frequencies, whereas those obtained using the time domain MFIE differ by as much as 10 dB from those obtained with FISC. Again, only “problematic” frequencies are shown in this and the next example. Similar observations can be deduced from Figure 2.14, which compares the RCS patterns in the xy-plane due to a \( \hat{k} = -\hat{x} \) propagating and \( \hat{p} = \hat{y} \) polarized Gaussian pulse with a
center frequency of $f_c = 200$ MHz and bandwidth $f_{bw} = 150$ MHz. It should be pointed out here that it is possible to extract meaningful results from the CFIE time domain data at 120 MHz in spite of the fact that at this frequency the power of the incident pulse is down by about 46 dB with respect to its peak value.

Figure 2.14: The RCS of an almond in the xy-plane, extracted from the time domain CFIE and MFIE is compared to that obtained from FISC for a set of frequencies. The incident wave is $\hat{p} = \hat{y}$ polarized and is traveling in the $\hat{k} = -\hat{x}$ direction.
2.5 Conclusions

In this chapter, the acoustic and electromagnetic surface scattering problems and the integral equation-based classical MOT schemes used in the solution of these problems were introduced. The resonance-suppressing properties of different time domain boundary integral equations for analyzing transient scattering were also investigated. It was shown that the solutions to both the Kirchhoff integral equation and its normal derivative for the acoustic case and the EFIE and MFIE for the electromagnetic case are vulnerable to corruption by cavity modes. For the acoustic case, it was also pointed out that the scattered fields calculated via the Kirchhoff integral equation are affected by the presence of interior modes, whereas those calculated through the normal derivative formulation are not. (The electromagnetic counterpart of this effect has been pointed out earlier in [74]). Time domain CFIEs were also introduced and it was shown both theoretically and experimentally that these integral equations effectively eradicate resonant components supported by the other time domain integral equations. Furthermore, it has been shown that the solution to the CFIE is accurate even at frequencies where the power in the incident pulse is very low. As will be clear in the next chapters, the proposed time domain CFIEs enable the accurate simulation of transient scattering from large structures, which possibly support cavity modes, using fast methods such as the PWTD algorithm.
CHAPTER 3
PLANE-WAVE TIME-DOMAIN ALGORITHMS FOR THE FAST ANALYSIS OF TRANSIENT WAVE PHENOMENA

3.1 Introduction

Evaluation of the scattered field due to past equivalent surface sources is computationally the most expensive part of classical marching-on-in-time (MOT) schemes, such as those presented in Sections 2.3.6 and 2.4.6. It is not only the MOT schemes that have suffered from the high computational cost of evaluating retarded time boundary integrals (RTBIs), which relate known transient source distributions to the fields they radiate. Exact nonreflecting boundary conditions, which can be used in truncating the computational domains of differential equation-based techniques, also rely on the evaluation of RTBIs. Thus, the high complexity of imposing these boundary conditions has outweighed their desirable characteristics rendering them unpopular.

Methods for evaluating the frequency domain counterparts of the RTBIs have also suffered from high computational complexities. However, many methods that significantly speed up the numerical evaluation of these boundary integrals have been proposed during the last decade. The fast multipole method (FMM) [49-53], the impedance matrix localization technique [54], and the multilevel matrix decomposition algorithm [55] can be named as examples of such methods. It is only through the use of these fast methods that tackling large-scale scattering and radiation problems using the method of moments has become feasible [53, 55] and imposing exact boundary conditions has become a viable approach [89]. Among these fast methods, FMM has enjoyed wide acceptance. This method relies on a plane wave representation of radiated wave fields. The main advantage of employing plane wave bases is that the translation operator
is diagonal. That is, the field tested throughout a bounded domain and represented in plane wave basis is related to its sources—also located inside a bounded domain remote from the observation domain and represented in plane wave basis—by a diagonal matrix.

This chapter introduces two PWTD algorithms that considerably reduce the cost of evaluating RTBIs by using plane wave bases. Strictly speaking, the introduced PWTD algorithms help evaluate wave fields that are radiated by a cluster of sources and that are tested at a group of observation points that are well separated from the source domain. The cost of evaluating the field due to an arbitrary surface source distribution—and hence, an MOT analysis—can be accelerated by partitioning the whole source distribution into small source clusters and by applying a PWTD algorithm within a two-level or a multilevel framework to calculate the fields due to each cluster. The first PWTD algorithm introduced in Section 3.3 is based on a Whittaker-type field expansion [2, 68, 69], i.e., a representation of the radiated field in terms of plane waves that propagate in all directions. To some extent, this scheme can be considered the direct time domain counterpart of the frequency domain FMM [49]. Using this PWTD algorithm in two-level and multilevel settings to accelerate MOT schemes reduces the $O(N_t N_s^2)$ computational complexity of a scattering analysis to $O(N_t N_s^{1.5} \log N_s)$ and $O(N_t N_s \log^2 N_s)$, respectively, where $N_t$ and $N_s$ denote the number of temporal and spatial degrees of freedom of the discretized surface equivalent sources. The second PWTD algorithm—introduced in Section 3.5 and henceforth referred to as the windowed PWTD algorithm—relies on a finite-cone representation in which only plane waves whose propagation directions fall within a cone encompassing the observation domain are used in representing the radiated field [6]. The computational costs of the PWTD-enhanced MOT algorithms based on the windowed PWTD algorithm can scale as low as $O(N_t N_s^{4/3} \log N_s)$ and $O(N_t N_s \log N_s)$. These algorithms can be considered the time domain analogues of the windowed fast multipole methods [51, 58, 90] and steepest descent path algorithms [91, 92] that have been so remarkably
successful in accelerating the solution of frequency domain scattering problems. Since the nature of the wave phenomenon being modeled and the boundary condition being enforced are secondary to the description of the PWTD schemes, the algorithms presented here are for a transient scalar wave solver that imposes a Dirichlet boundary condition on the scatterer. The generalization to different boundary conditions and to a Maxwell equation solver will be outlined in the following chapters.

This chapter is organized as follows. Section 3.2 outlines the classical MOT algorithm for analyzing scalar wave scattering from a scatterer characterized by a Dirichlet boundary condition. The nonwindowed PWTD scheme is presented in Section 3.3. Incorporation of this PWTD algorithm into the MOT scheme of Section 3.2 in two-level and multilevel settings is elucidated with the aid of graphical illustrations in Section 3.4. The resulting algorithms, which have computational complexities of $O(N_t N_s^{1.5} \log N_s)$ and $O(N_t N_s \log^2 N_s)$, are described in sufficient detail to permit the reader to easily retrofit existing MOT codes. Section 3.5 builds upon the ideas presented in Section 3.3 to present the windowed PWTD algorithm. The differences in incorporating the windowed PWTD into MOT schemes is highlighted and the computational complexities of the resulting algorithms are derived in Section 3.6. Finally, a summary of the chapter is given in Section 3.7.

3.2 The Marching-On-In-Time Method

Consider a field $u'(r, t)$ that is incident on a scatterer bounded by a surface $S$ as depicted in Figure 3.1. It is assumed that $u'(r, t)$ is temporally bandlimited to $\omega_{\text{max}}$. When $u'(r, t)$ interacts with $S$, a scattered field $u^s(r, t)$ is generated, such that the total field $u(r, t) = u'(r, t) + u^s(r, t)$ satisfies a given boundary condition on $S$. In this chapter, the derivations will be carried out assuming the homogeneous Dirichlet condition holds on $S$ to keep the exposition simple. However, as will be demonstrated in the following chapters, the
results of this chapter can be easily extended to cases with different boundary conditions. It is assumed that no interaction takes place before \( t = 0 \), i.e., \( u^S(r, t) = 0 \) for \( t < 0 \). Given that the total field satisfies the wave equation

\[
\nabla^2 u(r, t) - \frac{\partial^2}{c^2} u(r, t) = 0
\]

(3.1)
everywhere exterior to \( S \) and the radiation condition, it can be shown that \( u^S(r, t) \) can be represented in terms of equivalent surface sources \( q(r, t) \) that reside on \( S \) as

\[
u^S(r, t) = \int_S \frac{\delta(t - \frac{R}{c})}{4\pi R} q(r', t).
\]

(3.2)
In the above equation, \( c \) is the wave speed in the medium, \( \delta(\cdot) \) is the Dirac delta, \( R = |r - r'| \), and \( \ast \) denotes temporal convolution. The unknown source density \( q(r, t) \) in Eq. (3.2) can be related to the known incident field by enforcing the Dirichlet boundary condition \( u(r, t) = 0 \) on \( S \), which yields the integral equation

\[-u^i(r, t) = \int_S \frac{\delta(t - \frac{R}{c})}{4\pi R} q(r', t) \quad \forall r \in S.\]

(3.3)

Figure 3.1: The surface scattering problem. The incident field \( u^i(r, t) \) interacts with the scatterer bounded by the surface \( S \) and produces the scattered field \( u^S(r, t) \).
To solve Eq. (3.3) numerically, \( q(r, t) \) is represented in terms of spatial and temporal basis functions \( f_n(r), \; n = 1, \ldots, N_s, \) and \( T_i(t), \; i = 0, \ldots, N_t, \) as

\[
q(r, t) \cong \sum_{n=1}^{N_s} \sum_{i=0}^{N_t} q_{n,i} f_n(r) T_i(t), \tag{3.4}
\]

where the \( q_{n,i} \) represent unknown expansion coefficients. Choosing the number of basis functions as \( N_s \propto S_A(\omega_{\text{max}}/c)^2 \) and \( N_t \propto T\omega_{\text{max}} \), where \( S_A \) denotes the surface area of the scatterer and \( T \) the duration of the analysis, permits accurate representation of \( q(r, t) \). It is assumed that the spatial and temporal basis functions have local supports. Upon substituting Eq. (3.4) into Eq. (3.3), and testing the resulting equation at time \( t = t_j = j\Delta t \), where \( \Delta t \) is the time step size, with the testing functions \( \tilde{j}_m(r), \; m = 1, \ldots, N_s \), the following matrix equation is obtained:

\[
\mathbf{Z}_0 \mathbf{Q}_j = \mathbf{U}_j^j - \sum_{k=1}^{j-1} \mathbf{Z}_k \mathbf{Q}_{j-k}. \tag{3.5}
\]

In Eq. (3.5), the \( m^{\text{th}} \) elements of the vectors \( \mathbf{Q}_j \) and \( \mathbf{U}_j^j \) are given by \( q_{m,j} \) and 

\[- \int_S dr \; \tilde{j}_m(r) u^j(r, t_j) \], respectively, and the elements of the sparse interaction matrix \( \mathbf{Z}_k \) are given by

\[
\mathbf{Z}_{k, mn} = \int_S dr \; \tilde{j}_m(r) \int_S dr' f_n(r') \left[ \frac{\delta(t - R/c)}{4\pi R} \ast T_{j-k}(t) \right] \bigg|_{t=t_j}. \tag{3.6}
\]

Equation (3.5) relates the expansion coefficients at the \( j^{\text{th}} \) time step to the incident field and the coefficients at prior time steps. Hence, all the expansion coefficients \( q_{n,j} \) can be evaluated by starting at the first time step \( (j = 0) \), forming the right-hand side, and solving Eq. (3.4) at each time step. This procedure is known as the MOT scheme [3, 4, 31].
The evaluation of the summation appearing on the right-hand side of Eq. (3.5) is the most expensive operation in the MOT scheme. In essence, this operation entails the evaluation of the field \( u^S(r, t) \) at \( O(N_s) \) points on \( S \) due to all past sources, and requires \( O(N_s^2) \) operations. Since this operation is repeated for all \( N_t \) time steps in the analysis, the computational complexity of the classical MOT scheme scales as \( O(N_sN_t^2) \).

### 3.3 The Plane-Wave Time-Domain Algorithm

Evidently, the cost of the MOT scheme renders infeasible the practical analysis of scattering from large structures. Recent research efforts have focused on reducing this computational burden [78, 93-95]. This section will introduce a PWTD algorithm that achieves this goal by expressing transient fields in terms of plane waves. This algorithm permits the evaluation of transient fields at a set of observers when the sources and observers are clustered in well-separated regions in space. Also, as will be evident in the forthcoming analysis, the algorithm requires all source signals to be of limited temporal extent. However, this limitation does not hinder the applicability of the algorithm since the source signal \( q(r, t) \) can always be broken up into subsignals as

\[
q(r, t) = \sum_{l=1}^{L} q_l(r, t),
\]

where each subsignal \( q_l(r, t) \) is identically zero outside of an interval \( (l-1)T_s \leq t < lT_s \) as illustrated in Figure 3.2(a). Then, the field radiated by \( q_l(r, t) \), henceforth denoted as \( u_l(r, t) \), can be evaluated using the PWTD algorithm, and the total field formed as

\[
u(r, t) = \sum_{l=1}^{L} u_l(r, t).
\]

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Figure 3.2: Sectioning of a signal into subsignals of duration $T_s$ (a) such that $l^{th}$ subsignal is zero outside the interval $(l-1)T_s \leq t < lT_s$, (b) such that each subsignal is bandlimited to $\omega_s > \omega_{\text{max}}$.

3.3.1 Plane wave decomposition

Motivated by frequency domain fast multipole schemes [49-51, 90], which reconstruct the time harmonic field due to a group of sources as a superposition of plane waves, our first task is to postulate a suitable plane wave representation for transient wave fields. Such representations date back to Whittaker [68] and have since been thoroughly studied [2, 6, 67, 69, 96]. To arrive at a plane wave representation that facilitates the fast evaluation of transient fields, consider the expression

$$\tilde{u}_l(r,t) = \frac{-\partial_t}{8\pi^2c} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \int_S dr' \delta[t - \hat{k} \cdot (r - r')/c] \ast q_l(r',t).$$  \hspace{1cm} (3.9)

In Eq. (3.9), $\hat{k} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$ is a unit direction vector and the innermost integral over $S$ is a projection of the source distribution $q_l(r,t)$ onto a plane wave travelling in the $\hat{k}$ direction. Hence, $\tilde{u}_l(r,t)$ is a superposition of plane waves that travel in all directions.
Carrying out the integrations in Eq. (3.9) reveals an interesting relation between \( \tilde{u}_l(r,t) \) and \( u_l(r,t) \). To this end, the spherical integral in Eq. (3.9) is cast in a new coordinate system \((x', y', z') = (\rho', \theta', \phi')\) in which the \( z' \) axis is aligned with \( \mathbf{R} = r - r' \). In this new coordinate system \( \hat{\mathbf{k}} \cdot \mathbf{R} = R \cos \theta' \). Then, a simple change of variables of \( \tau = R \cos \theta' / c \) allows one to relate \( \tilde{u}_l(r,t) \) to \( u_l(r,t) \) as follows:

\[
\tilde{u}_l(r,t) = -\frac{\hat{d}_l}{8\pi^2 c} \int_S \int d\phi' \int_0^\pi d\theta' \sin \theta' \delta(t - R \cos \theta'/c) \ast q_l(r',t)
\]

\[
= -\frac{\hat{d}_l}{8\pi^2 c} \int_S \int d\phi' \int_0^\pi d\theta' \sin \theta' \delta(t - R \cos \theta'/c) \ast q_l(r',t)
\]

\[
= - \int_{-R/c}^{R/c} \int_S d\tau \frac{\hat{d}_l \delta(t-\tau)}{4\pi R} \ast q_l(r',t)
\]

\[
= \int_S \frac{\hat{d}_l \delta(t-R/c)}{4\pi R} \ast q_l(r',t) - \int_S \frac{\hat{d}_l \delta(t+R/c)}{4\pi R} \ast q_l(r',t)
\]

\[
= u_l(r,t) - \int_S \frac{\hat{d}_l \delta(t+R/c)}{4\pi R} \ast q_l(r',t).
\] (3.10)

Were it not for the second term appearing on the right-hand side, \( \tilde{u}_l(r,t) \) would equal \( u_l(r,t) \)!

The second term is referred to as the ghost signal and is anticausal, i.e., it appears at the observer before the source signal exists, as shown in Figure 3.3. Clearly, if the signal duration \( T_s \) is shorter than \( R/c \), the source signal \( q_l(r,t) \) vanishes before any true field \( u_l(r,t) \) appears at the observer. This condition also ensures that the true signal and ghost signal never overlap. Thus, choosing \( T_s < R/c \) as the subsignal duration permits one to time-gate out the ghost signal. In [2], Tygel and Hubral introduced a similar mechanism to avoid the ghost signal, which they refer to as the “causality trick.”

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Figure 3.3: Naïve application of Eq. (3.9) to a source signal results in the true observed field and an anticausal ghost field.

Figure 3.4: Definitions of the vectors used in the three-stage PWTD algorithm.

To explore the merits of the field representation provided by Eq. (3.9) in the construction of a fast algorithm for evaluating transient fields, consider a source distribution that is confined to a sphere of radius $R_s$, and a set of observers that are located inside a sphere of equal radius (Figure 3.4). The centers of the source and observer spheres are denoted by $r_s$ and $r_o$, respectively, and the vector connecting the respective sphere centers by $R_c = r_o - r_s$. It is assumed that $R_c = |R_c| > 2R_s$, i.e., the source and observer spheres do not overlap. Next, noting that the vector $r - r'$ can be decomposed as $r - r' = (r - r_o) - R_c - (r' - r_s)$, Eq. (3.9) can be rewritten as

$$
\tilde{u}(r,t) = \int d^2 k \delta \left[ t - \hat{k} \cdot (r - r_o) / c \right] * T(\hat{k}, R_c, t) * \int_{S} dr' \delta \left[ t + \hat{k} \cdot (r' - r_s) / c \right] * q_i(r', t), \quad (3.11)
$$

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where \( \int d^2\hat{k} = \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \) denotes integration over the unit sphere and the translation function \( T(\hat{k}, R_c, t) \) is given by

\[
T(\hat{k}, R_c, t) = -\frac{\hat{k}_r}{8\pi^2 c} \delta(t - \hat{k} \cdot R_c / c).
\] (3.12)

Next, an algorithm that evaluates \( \tilde{u}_f(r, t) \) via a three-stage implementation of Eq. (3.11) will be outlined, and the conditions for the reconstruction of \( u_f(r, t) \) from \( \tilde{u}_f(r, t) \) by proper time gating will be discussed. The scheme proceeds as follows.

1. Perform the rightmost convolution and integration in Eq. (3.11), i.e., evaluate

\[
q_i^{\text{out}}(\hat{k}, t) = \int_\mathcal{S} dr' \delta \left[ t + \hat{k} \cdot (r' - r_s)/c \right] \ast q_i(r, t),
\] (3.13)

which is known as the slant stack transform (SST) of the source distribution \( q_i(r, t) \) [6, 8, 97]. The quantities \( q_i^{\text{out}}(\hat{k}, t) \) can be interpreted as outgoing rays, i.e., rays that leave the source sphere in the direction \( \hat{k} \). All \( q_i^{\text{out}}(\hat{k}, t) \) together provide the source’s transient far-field pattern. Note that \( q_i^{\text{out}}(\hat{k}, t) \) is completely characterized as soon as \( q_i(r, t) \) ceases to radiate, i.e., at \( t = IT_s \).

2. Carry out the center convolution in Eq. (3.11), i.e., evaluate

\[
q_i^{\text{in}}(\hat{k}, t) = T(\hat{k}, R_c, t) \ast q_i^{\text{out}}(\hat{k}, t).
\] (3.14)

The operator \( T(\hat{k}, R_c, t) \ast \) translates each outgoing ray from the source sphere to the observer sphere. The quantities \( q_i^{\text{in}}(\hat{k}, t) \) are termed incoming rays, i.e., rays that impinge upon the observation sphere. Since this operation maps an outgoing ray to an incoming ray that travels in the same direction, \( T(\hat{k}, R_c, t) \ast \) can be considered a diagonal translation operator for transient scalar wave fields.
3. Perform the leftmost integration and convolution in Eq. (3.11), i.e., evaluate
\[
\tilde{u}(r, t) = \int d^2 \hat{\mathbf{k}} \delta \left[ t - \hat{\mathbf{k}} \cdot (r - r_0)/c \right] \ast q^{jn}(\hat{\mathbf{k}}, t).
\]  
(3.15)

This operation superimposes the projections of all the incoming rays onto the observer.

The signals associated with this three-stage algorithm are depicted in Figure 3.5. Note that the observed field depicted in Figure 3.5(g) consists of a ghost signal and the true field as implied by Eq. (3.10).

![Diagram](image)

Figure 3.5: The signals associated with the three-stage PWTD algorithm. (a) The source-observation configuration and four selected ray directions. (b) The source signal. (c) The outgoing rays propagating along the selected ray directions. Note that for any source location, the outgoing rays are confined to the interval \([−R_s, R_s + cT_s]\). (d) The \(\delta(t - \mathbf{k} \cdot \mathbf{R}_c/c)\) term of the translation functions associated with the selected ray directions. (e) Incoming rays obtained after translation operation. (f) Projection of the incoming rays onto the observer location. (g) The observed field.

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Figure 3.5: (continued).
As was shown earlier, the distance between the source and observation points imposes an upper bound on the subsignal duration. Consequently, considering the nearest possible source and observation locations in the two-sphere setting, the choice of \( T_s < (R_c - 2R_s)/c \) permits the elimination of the ghost signal with proper time gating. This condition implies that \( u_l(r, t) = 0 \) for \( t < IT_s \) as a time span of at least \((R_c - 2R_s)/c\) should elapse from the onset of the source subsignal \( q_l(r, t) \) at \( t = (l-1)T_s \) for any field to propagate to the observer sphere. Also, \( u_l(r, t) = \bar{u}_l(r, t) \) for \( t \geq IT_s \) since the ghost signals vanish throughout the observer sphere for \( t \geq -(R_c - 2R_s)/c + IT_s \) by Eq. (3.10) and the observer and source spheres are assumed to be non-overlapping, i.e., \( R_c > 2R_s \). Therefore, choosing \( T_s < (R_c - 2R_s)/c \) ensures that

\[
  u_l(r, t) = \begin{cases} 
  0 & \text{for } t < IT_s \\
  \bar{u}_l(r, t) & \text{for } t \geq IT_s 
  \end{cases}
\]  
(3.16)

### 3.3.2 Implementation issues

Several remarks regarding the implementation of the three-stage PWTD algorithm are in order:

i) **Spatial integration.** The integration over the source domain in Eq. (3.13) should be carried out using appropriate quadrature rules. The same holds true when testing the fields, as in Eq. (3.6). Since the choice of integration rule depends on the actual discretization function used for representing the source distribution or the testing function, this issue will not be further discussed here.

ii) **Spectral integration.** To numerically evaluate the fields via Eq. (3.11), the integration over the unit sphere has to be performed using an appropriate quadrature rule that requires sampling of the integrand only in a small number of directions. Three basic observations allow us achieve arbitrary accuracy when numerically evaluating Eq. (3.11). First,
the integrand in Eq. (3.11), excluding the translation function, can be interpreted as the time dependent radiation pattern of a source distribution enclosed in a sphere of radius $2R_s$ [8]. Therefore, assuming that $q_l(r, t)$ is temporally bandlimited to some $\omega_s > \omega_{\text{max}}$ (see item iii) below), this part of the integrand can be represented in terms of spherical harmonics $Y_{km}(\theta, \phi)$ as

$$
g(\hat{k}, r, t) = \delta[t - \hat{k} \cdot (r - r_o)/c] \ast \int _S \delta[t + \hat{k} \cdot (r' - r_s)/c] \ast q_l(r', t)
= \int _S \delta[t + \hat{k} \cdot [(r' - r_s) - (r - r_o)]/c] \ast q_l(r', t)
= \sum _{k=0} ^{K} \sum _{m=-k} ^{k} g_{km}(r, t) Y_{km}(\theta, \phi).
$$

(3.17)

where $K = \lfloor \chi_1 2R_s \omega_s / c \rfloor$ and $\chi_1 > 1$ is an excess bandwidth factor [98, 99] that ensures rapid convergence of the series in Eq. (3.17). Secondly, the translation function is only a function of the angle $\theta'$ between $\hat{k}$ and $R_c$, and can be expressed in terms of associated Legendre polynomials in $\theta'$ or the spherical harmonics in $(\theta, \phi)$ as

$$
T(\hat{k}, R_c, t) = \begin{cases} 
\frac{\partial_t}{16\pi^2 R_c} \sum _{k=0} ^{\infty} (2k+1)P_k(ct/R_c)P_k(\cos \theta') & ; |t| \leq R_c/c \\
0 & ; \text{elsewhere}
\end{cases}
$$

(3.18)

Finally, due to the orthogonality of the spherical harmonics, the terms in Eq. (3.18), for which $k > K$, do not contribute to the final result when integrating $g(\hat{k}, r, t) \ast T(\hat{k}, R_c, t)$ over the unit sphere. Hence, the summation in Eq. (3.18) can be truncated at $k = K$, and, in all previous expressions, the translation function $T(\hat{k}, R_c, t)$ can be replaced by its truncated version $\widetilde{T}(\hat{k}, R_c, t)$ given by
\[ \mathcal{F}(\mathbf{k}, R_c, t) = \begin{cases} \frac{\partial_t}{16 \pi^2 R_c^2} \sum_{k=0}^{K} (2k + 1) P_k(ct/R_c) P_k(\cos \theta') & ; |t| \leq R_c/c \\ 0 & ; \text{elsewhere} \end{cases} \] (3.19)

As illustrated in Figure 3.6, with this truncation, the translation function becomes nonlocalized and spans a duration of \(2R_c/c\). Note that, as the harmonics with \(k > K\) do not contribute to the final result, they can be tapered off with a smooth windowing instead of the hard truncation in Eq. (3.19) so that the translation function spans a shorter duration. This also implies that the translation function is not unique. This point will be further exploited in Sections 3.5 and 3.6 to develop more efficient algorithms.

![Figure 3.6: Effect of bandlimiting the translation function. The lower part illustrates the Dirac functions expressed by the infinite series in Eq. (3.18) and the upper part the truncated series of Eq. (3.19) for \(K = 30\).](image)

It is now clear that the integrand in Eq. (3.11) can be written as a product of two functions, each of which is expressible in terms of spherical harmonics \(Y_{km}(\theta, \phi), \ k = 0, \ldots, K\).
Such an integral can be evaluated exactly by using a \((2K + 1)\)-point trapezoidal rule in the \(\phi\) direction and a \((K + 1)\)-point Gauss-Legendre quadrature in the \(\theta\) direction yielding the following expression for \(\tilde{u}_i(r, t)\):

\[
\tilde{u}_i(r, t) = \sum_{p=0}^{K} \sum_{q=-K}^{K} w_{pq} \delta\left[t - \hat{k}_{pq} \cdot (r - r_o)/c\right] \ast \tilde{F}(\hat{k}_{pq}, R_c, t) \ast 
\int_{S} dr' \delta\left[t + \hat{k}_{pq} \cdot (r' - r_s)/c\right] \ast q_l(r', t),
\]

where

\[
\begin{align*}
\hat{k}_{pq} &= \hat{x} \sin \theta_p \cos \phi_q + \hat{y} \sin \theta_p \sin \phi_q + \hat{z} \cos \theta_p, \\
\phi_q &= q2\pi/(2K + 1), \\
\theta_p &= \text{the} (p + 1)\text{th zero of } P_{K+1}(\cos \theta).
\end{align*}
\]

Although this choice of sampling points seems to depend on the coordinate system, this is not true owing to the fact that under a coordinate system rotation a spherical harmonic of degree \(K\) is transformed into a linear combination of harmonics of the same degree [100, 101]. Hence, as already implied by the second equality in Eq. (3.18), sampling points can be chosen with respect to any coordinate system independent of \(R_c\).

iii) **Subsignal temporal representation.** Finally, in the derivation of the above quadrature rule, it was assumed that the subsignals are temporally bandlimited. This assumption contradicts the requirement that each subsignal has to be time limited as well. However, the whole signal that is bandlimited to \(\omega_{\text{max}}\) can be broken up into subsignals that are both bandlimited to \(\omega_s = \chi_2 \omega_{\text{max}}\) with \(\chi_2 > 2\) and approximately time limited to a given interval length by using proper local interpolation functions as described in detail in Section 3.5.2. The
sectioning of a signal using these interpolants yields overlapping subsignals as illustrated in Figure 3.2(b). It can be shown that the error introduced by using subsignals thus generated can be decreased to any desired precision by increasing $\chi_2$. Hence, the accuracy of the three-stage PWTD algorithm is solely determined by the choice of $\chi_1$ and $\chi_2$.

To verify this and to determine the level of accuracy that can be achieved using the proposed algorithm, several numerical experiments were conducted. The results of one such experiment are depicted in Figure 3.7. In this experiment, eight point sources were distributed

Figure 3.7: Energy norm of the error calculated by the three-stage PWTD algorithm around the observation sphere. The error estimates prior to the calculation were of order (a) $10^{-4}$, (b) $10^{-7}$, (c) $10^{-10}$, (d) $10^{-14}$.
over a unit sphere centered at \((x,y,z) = (-5,-5,-5)\) meters and an observation sphere of unit radius was centered at the origin. The time signature of each source was a Gaussian pulse given by \(e^{-(t-7.75\sigma)^2/(2\sigma^2)}\) with \(\sigma = 3.183\) ns and the wave speed of the medium was assumed to be \(c = 3 \times 10^8\) m/s. The radiated fields were evaluated via the three-stage PWTD algorithm throughout a \(6 \times 6\) m region in the \(x = y\) plane centered around the observation sphere. Then, the calculated fields were compared with those obtained analytically. The energy norm of the error thus obtained is plotted in Figure 3.7 for four cases, in which the parameters used for forming the subsignals as well as the truncation limit for the translation functions were estimated to yield errors of order \(10^{-4}\), \(10^{-7}\), \(10^{-10}\), and \(10^{-14}\). As seen in the figure, the targeted level of accuracy has been achieved within the observer sphere for each case.

3.4 Implementation of the PWTD Enhanced MOT Schemes

The previous section described a PWTD algorithm that permits the plane wave-based evaluation of transient fields due to a known source distribution. Incorporation of this algorithm into the classical MOT framework results in source reconstruction schemes whose costs scale much more favorably than that of the conventional MOT method. The most naïve way to combine the PWTD and MOT schemes is to divide the scatterer into equally sized subscatterers and to evaluate interactions between nearby and distant subscatterers using the classical MOT and PWTD algorithms, respectively. This yields a “two-level” scheme with \(O(N_s N_{s1}^{1.5} \log N_s)\) complexity. A more efficient “multilevel” algorithm with \(O(N_t N_s \log^2 N_s)\) complexity is obtained by grouping very distant subscatterers into even larger entities before invoking the PWTD scheme (Figure 3.8). In this section, the two-level algorithm will be described first and its multilevel extension will be outlined next.
3.4.1 A two-level PWTD enhanced MOT algorithm

The basic idea supporting the two-level PWTD enhanced MOT scheme is very simple. To rapidly evaluate the sum appearing on the right-hand side of Eq. (3.5), the scatterer is subdivided into a large number of subscatterers. Next, all contributions to this sum that arise from spatial basis and testing functions residing on nearby subscatterers are evaluated directly, i.e., as in Eq. (3.5). All other contributions are evaluated using the PWTD scheme.

These ideas are formalized using the following definitions. Consider a fictitious cubical volume enclosing the scatterer that is subdivided into many equally sized smaller boxes, each of which fits into a circumscribing sphere of radius $R_s$ as illustrated in Figure 3.8(a). Let $N_g$
denote the total number of nonempty boxes. Henceforth, the collection of spatial basis functions that reside within a nonempty box is termed a group and the average number of spatial basis functions per group is denoted by $M_s = N_s / N_g$. It can be verified that, for a surface scatterer, $M_s \propto (R_s \omega_{\max} / c)^2$. Each group pair $(\gamma, \gamma')$, $\gamma, \gamma' = 1, \ldots, N_g$, is identified as either a “near-field” or a “far-field” pair depending on whether their group centers are separated by less than or more than a preset distance $R_{c, \text{min}}$, respectively. It is assumed that $R_{c, \text{min}} = \xi R_s$, where $\xi$ typically varies between 3 and 6. This implies that every group participates in only a small number of near-field pairs, and therefore that the total number of near-field pairs is proportional to $N_g$.

In order to facilitate a ghost-free evaluation of fields involving far-field source and observer groups using the PWTD algorithm, appropriate subsignal durations must be defined. To this end, the fundamental subsignal duration $T_s$ is defined as $T_s = M_t \Delta t$, where

$$M_t = \min_{\gamma, \gamma'} \left\{ \left\lfloor \frac{(R_{c, \gamma'} - 2 R_s) / (c \Delta t)}{R_{c, \gamma}} \right\rfloor \right\}$$

and $R_{c, \gamma}$ denotes the distance between the centers of groups $\gamma$ and $\gamma'$. This $T_s$ is easily identified as the maximum duration of a subsignal that can be translated without incurring any ghost signals between the nearest groups classified as a far-field pair. Subsignal durations that feature in the evaluation of fields related to other far-field pairs $(\gamma, \gamma')$ are represented as $T_{s, \gamma \gamma'} = M_{t, \gamma \gamma'} \Delta t$. The integers $M_{t, \gamma \gamma'}$ are multiples of $M_t$ and are computed as $M_{t, \gamma \gamma'} = M_t \left\lfloor \left( R_{c, \gamma \gamma'} - 2 R_s \right) / (c \Delta t) \right\rfloor$; hence, all subsignal durations $T_{s, \gamma \gamma'}$ are integer multiples of the fundamental subsignal duration $T_s$. Note that these definitions enable one to recycle information since outgoing rays that feature in the interaction of any far-field pair can be obtained by concatenating an integer number of rays corresponding to subsignals of duration $T_s$.

Before embarking on a systematic description of the two-level scheme, a few words are in order regarding the convolution of outgoing rays with translation functions (Eq. (3.14)). The
maximum duration of the outgoing rays that are to be translated between groups \(\gamma\) and \(\gamma'\) is \(T_{s,\gamma\gamma'} + 2R_s\), which can be shown to be proportional to \(R_{c,\gamma\gamma'}/c \propto M_{t,\gamma\gamma'} \Delta_t\) with the aid of the above definitions. Also, as evident from Eq. (3.19), the duration of the translation functions is proportional to \(R_{c,\gamma\gamma'}/c\) — in fact, it exactly equals \(2R_{c,\gamma\gamma'}/c\). This implies that both signals that are to be convolved last \(O(M_{t,\gamma\gamma'})\) time steps. To evaluate this convolution using fast Fourier transform (FFT) techniques in \(O(M_{t,\gamma\gamma'} \log M_{t,\gamma\gamma'})\) operations, note that, as mentioned in Section 3.3.2, the subsignals and therefore the outgoing rays are bandlimited. Consequently, their spectrum can be obtained using FFTs. However, as the translation functions are not bandlimited, their spectrum cannot be obtained by simple FFTs. Fortunately, analytic expressions for the Fourier transforms of the translation functions are available, enabling the rapid evaluation of their spectra. Using Eq. (3.19), the Fourier transform of the translation functions is

\[
\mathcal{F}\{\tilde{\mathcal{F}}(\mathbf{k}, R_{c,\gamma\gamma'}, t)\} = \int_{-\infty}^{+\infty} dt \tilde{\mathcal{F}}(\mathbf{k}, R_{c,\gamma\gamma'}, t) e^{-j\omega t}
\]

\[
= -\frac{j\omega}{8\pi^2c} \sum_{k=0}^{K} (2k+1)(-j)^k j_k\left(\frac{\omega R_{c,\gamma\gamma'}/c}{\omega R_{c,\gamma\gamma'}/c}\right) P_k(\cos \theta'),
\]

where \(j_k(\cdot)\) is the spherical Bessel function of order \(k\). An efficient way to construct these functions for any sphere pair \((\gamma, \gamma')\) is to tabulate the normalized translation function

\[
\tilde{\mathcal{F}}(\theta', \Omega) = R_{c,\gamma\gamma'} \mathcal{F}\{\tilde{\mathcal{F}}(\mathbf{k}, R_{c,\gamma\gamma'}, t)\}
\]

\[
= -\frac{j\Omega}{8\pi^2} \sum_{k=0}^{K} (2k+1)(-j)^k j_k(\Omega) P_k(\cos \theta')
\]

with respect to the ray angle \(\theta'\) and the normalized frequency \(\Omega = \omega R_{c,\gamma\gamma'}/c\). Since \(\tilde{\mathcal{F}}(\theta', \Omega)\) is bandlimited in both \(\theta'\) and \(\Omega\), it can be reconstructed from judiciously selected samples in the \((\theta', \Omega)\) plane. Specifically, it can be shown that \(O(\max_{\gamma,\gamma'}\{M_{t,\gamma\gamma'}\}) = O(\sqrt{N_s})\) samples in \(\Omega\)
and $O(K)$ samples in $\theta'$ permit the reconstruction of $\tilde{F}(\theta', \Omega)$ to arbitrary precision for all far-field pairs.

A two-level algorithm for the rapid evaluation of the sum on the right-hand side of Eq. (3.5) can now be prescribed. In this algorithm, the contributions to this sum due to all near- and far-field pairs are evaluated separately.

1. **Evaluation of the near-field contributions.** At each time step, the sum

$$
\sum_{k=1}^{j-1} \bar{Z}_{k}^{\gamma'} \bar{Q}_{j-k}^{\gamma'}
$$

is computed for all near-field pairs $(\gamma, \gamma')$. In Eq. (3.24), $\bar{Z}_{k}^{\gamma'}$ denotes the submatrix of the interaction matrix $\bar{Z}_{k}$ that relates fields over group $\gamma$ to sources residing in group $\gamma'$, and $\bar{Q}_{j-k}^{\gamma'}$ is a vector comprised of the $q_{n,j-k}$ for all sources $n$ in group $\gamma'$. For every time step, the evaluation of the sum (3.24) requires $O(M_{s}^{2})$ operations per near-field pair. Since each group interacts in a direct fashion only with a small set of near-field neighbors, the overall cost of this step scales as $O(N_{s}N_{g}M_{s}^{2}) = O(N_{s}N_{s}M_{s})$.

2. **Evaluation of the far-field contributions.** The three-step procedure outlined in Section 3.3.1 that accounts for the far-field contributions is incorporated into the MOT scheme as follows.

a) **Construction of outgoing rays.** For each group, a set of outgoing rays describing transient far-fields that are generated by subsignals of fundamental duration $T_{s}$ is constructed every $M_{t}$ time steps. Each such set consists of $O(K^{2}) = O(R_{s}^{2} \omega_{\text{max}}^{2} / c^{2}) = O(M_{s})$ rays that are formed by contributions from all sources residing in a group. The contribution from the spatial basis function $f_{n}(r)$ to the outgoing ray propagating in the $\hat{k}_{pq}$ direction is obtained by convolving the subsignal associated with $f_{n}(r)$ with
\[ V_n^+(\mathbf{k}_{pq}, t) = \int_\mathcal{S} \delta \left[ t + \mathbf{k}_{pq} \cdot (\mathbf{r}' - \mathbf{r}_c) / c \right] f_n(\mathbf{r}'), \] (3.25)

where \( \mathbf{r}_c \) denotes the center of the group to which \( f_n(\mathbf{r}) \) belongs. Since this mapping from \( O(M_s) \) sources to \( O(K^2) \) rays is carried out for all \( N_g \) groups a total of \( N_t / M_t \) times and since the cost of projecting a single spatial source onto a single direction scales as \( O(M_t) \) per subsignal, the cost of constructing all outgoing rays for the duration of the analysis is \( O(M_s K^2 N_g (N_t / M_t) M_t) = O(N_t N_s M_s) \).

b) Translation. For each far-field pair \((\gamma, \gamma')\), outgoing rays are translated (i.e., converted from outgoing to incoming rays) from group \( \gamma \) to \( \gamma' \) every \( M_{t,\gamma'} \) time steps. Note that the duration of the subsignals that are translated is \( T_{s,\gamma'} \). As mentioned previously, although outgoing rays describing such subsignals are not readily available, they can be formed by concatenating \( M_{t,\gamma'}/M_t \) rays of fundamental duration \( T_s \) computed in Step 2a. This is feasible as \( M_{t,\gamma'} \) is an integer multiple of \( M_t \) by design. Each outgoing ray is translated to an incoming ray by executing the following steps:

i. An outgoing ray described by \( O(M_{t,\gamma'}) \) time samples is constructed by concatenating the rays stored in Step 2a. Furthermore, anticipating its convolution with a translation function, its Fourier transform is computed using an FFT, which can be accomplished in \( O(M_{t,\gamma'} \log M_{t,\gamma'}) \) operations.

ii. The spectrum of the pertinent translation function is extracted from the \( \tilde{T}(\theta',\Omega) \) table through local interpolation in \( O(M_{t,\gamma'}) \) operations.

iii. The outgoing ray spectrum is multiplied with the translation function spectrum in \( O(M_{t,\gamma'}) \) operations.

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iv. The result is inverse Fourier transformed back into time domain in
$O(M_{t,\gamma'} \log M_{t,\gamma'})$ operations. The rays obtained through this procedure are
superimposed onto incoming rays that impinge upon the receiver group from all other
source groups.

Clearly, the dominant cost per direction scales as $O(M_{t,\gamma'} \log M_{t,\gamma'})$. For a single
group pair, the above steps are to be repeated $N_t/M_{t,\gamma'}$ times to cover the duration of
the entire transient analysis, and for all $O(K^2)$ directions, which results in a cost that
scales as $O((N_t/M_{t,\gamma'})K^2 M_{t,\gamma'} \log M_{t,\gamma'}) = O(N_t M_s \log M_{t,\gamma'})$. Note that the
subsignal duration $M_{t,\gamma'}$ is proportional to the distance between groups, which is
bounded by the maximum linear dimension of the scatterer. For a surface scatterer, the
maximum linear dimension is proportional to $\sqrt{N_s}$. Hence, in the above estimate,
$\log M_{t,\gamma'}$ can safely be replaced by $\log N_s$. The total computational cost of translation
between all $O(N_g^2)$ far-field group pairs therefore scales as $O(N_g^2 N_t M_s \log N_s) =
O(N_t N_g^2 M_s^{-1} \log N_s)$.

c) **Projection of incoming rays onto observers.** At each time step, the field at the $n^{th}$
other is formed by convolving the incoming rays with

$$
\tilde{v}_n(\hat{k}_{pq}, t) = \int_{S} dr' \delta[t - \hat{k}_{pq} \cdot (r' - r_c)/c] \hat{f}_n(r'),
$$

(3.26)

and by performing the spherical integration, i.e., adding up the signals due to all
incoming rays. This step constitutes a mapping from $O(K^2)$ directions to $O(M_s)$
resources for all groups and all time steps, and can be accomplished in $O(N_t N_g K^2 M_s) =
O(N_t N_g M_s)$ operations.

Evidently, the computational complexity of each of the steps in this algorithm depends on
the group size $M_s$. It can be verified by adding up the costs associated with steps 1 and 2 above
that the overall complexity is minimized by choosing $M_s$ proportional to $\sqrt{N_s}$. With this choice, the computational cost of performing a transient scattering analysis using the two-level algorithm scales as $O(N_f N_s^{1.5} \log N_s)$.

3.4.2 A multilevel PWTD enhanced MOT algorithm

A multilevel PWTD-enhanced MOT scheme can be constructed by casting the two-level algorithm into a divide-and-conquer framework. The procedure that permits a further reduction in computational complexity of the two-level scheme involves the multilevel aggregation of small subscatterers into larger entities before translation, inasmuch as permitted by the PWTD algorithm. This section will introduce a systematic scheme for achieving a multilevel partitioning of a scatterer along with an arsenal of multilevel notation. Also, four operations—ray interpolation, ray splicing, ray resection, and ray anterpolation—that permit the transfer of information across levels will be elucidated. Finally, the multilevel PWTD algorithm will be described in detail.

A hierarchical subdivision of the scatterer is achieved by recursively subdividing a fictitious cubical box that encloses the scatterer. Initially, this box is divided into eight boxes, each of which is further subdivided into eight smaller boxes in a recursive manner. A box that is subdivided into smaller boxes is termed the “parent” of the “child” boxes that result from the operation. The finest boxes so obtained are termed level 1 boxes and the collection of spatial basis functions that fall into a level 1 box is said to form a level 1 group; higher level boxes and groups are defined similarly. For levels $i = 1, \ldots, N_f$, let $N_g(i)$ denote the number of groups (nonempty boxes), $M_s(i)$ the average number of sources in each group, $R_s(i)$ the radius of the sphere that encloses a level $i$ box, and $K(i)$ the number of spherical harmonics used in construction of the translation functions. It then follows from the discussion of the two-level
scheme that $M_s(i) \propto (R_s(i) \omega_{\text{max}}/c)^2 \propto K^2(i)$. It is assumed that level 1 boxes have linear dimensions that are proportional to the wavelength at $\omega_{\text{max}}$, i.e., $(R_s(1) \omega_{\text{max}}/c)$ is of $O(1)$. This in turn implies that level 1 groups contain only a small number of basis functions that is independent of the problem size, i.e., $M_s(1)$ is of $O(1)$. It also follows that $N_g(1) \propto N_s$ and that $N_l \propto \log N_s$. Finally, note that, for a surface scatterer, $N_g(i+1) \propto N_g(i)/4$ and $M_s(i+1) \propto 4M_s(i)$, and therefore $N_g(i+1)M_s(i+1) \approx N_g(i)M_s(i) \approx N_g(1)M_s(1) \approx N_s$.

Next, a set of near- and far-field group pairs is constructed. Just like in the two-level algorithm, each and every source/observer (basis/testing function) combination belongs to one and only one pair. However, in contrast to the two-level PWTD algorithm, all far-field pairs do not all reside at the same level: distant source/observer combinations tend to belong to higher-level far-field pairs than those that reside close to one another (Figure 3.8(b)). Consequently, cutoff separations for all levels are defined as $R_{c,\text{min}}(i) = \xi R_s(i)$. To construct the near- and far-field pairs, first, each group pair at the highest level whose centers are separated by more than $R_{c,\text{min}}(N_l)$ is classified as a "level $N_l$ far-field pair." Next, all level $N_l - 1$ pairs with group centers separated by more than $R_{c,\text{min}}(N_l - 1)$, and describing interactions that have not yet been accounted for by any of the level $N_l$ far-field group pairs, are classified as "level $N_l - 1$ far-field pairs." This process is continued, and far-field pairs are identified at each and every level, including level 1, as those pairs that are considered well separated at a given level, but that have not yet been accounted for at a higher level. The level 1 pairs with group centers separated by less than $R_{c,\text{min}}(1)$ are classified as near-field pairs.

As in the two-level algorithm, the fundamental subsignal duration for level $i$ is calculated as $T_s(i) = M_f(i) \Delta_f$, with $M_f(i) = \min_{\gamma, \gamma'} \left[ \left[ (R_{c,\gamma'} - 2R_s(i))/(c \Delta_f) \right] \right]$ where $\gamma$ and $\gamma'$ vary over all the level $i$ far-field groups. Also, subsignal durations for evaluation of fields involving a far-
field pair \((\gamma, \gamma')\) at level \(i\) are defined as \(T_{S,\gamma\gamma'}(i) = M_{t,\gamma\gamma'}(i)\Delta_t\) with \(M_{t,\gamma\gamma'}(i) = M_t(i)\left[(R_{c,\gamma\gamma'} - 2R_s(i))/(cT_s(i))\right]\).

For a given level \(i\) far-field group pair \((\gamma, \gamma')\), the field at group \(\gamma\) due to \(q_t(r,t)\) in group \(\gamma'\) (and vice versa) will be calculated using the three-stage PWTD algorithm. The first stage of this algorithm calls for the evaluation of the SSTs to form the outgoing rays associated with group \(\gamma'\) along \(O(K^2(i))\) directions. For \(i=1\), the outgoing rays can be obtained by directly evaluating \(V_n^+ (\hat{k}_{pq}, t)\) as in Eq. (3.25). However, for higher levels, direct evaluation of the SSTs is computationally expensive. The fact that the same source information is to be used to construct outgoing rays at each level suggests that rays at level \(i > 1\) can be constructed economically by reusing information already stored in level \(i-1\) rays. This is achieved by two operations termed interpolation and splicing, which are schematically illustrated in Figure 3.9(a). Interpolation is needed as more rays are to be associated with a parent group than with one of its children and splicing is used in assembling a single parent group ray from interpolated child rays. Similarly, the last stage of the PWTD algorithm calls for a projection of incoming rays onto observers. Directly projecting the incoming rays at level \(i > 1\) onto the observers is more expensive than disaggregating these rays into level \(i - 1\) rays and propagating the information contained in these rays through the multilevel structure until level 1 rays are projected onto the observer locations using Eq. (3.26). Two operations termed resection and anterpolation, which are complementary to the splicing and interpolation operators, construct level \(i - 1\) incoming rays for a group from those of its parent as illustrated in Figure 3.9(b). Implementation of these four operations is discussed next.
Figure 3.9: (a) Ray interpolation and ray splicing operations to form the outgoing rays of level \( i+1 \) from the rays of level \( i \). (b) Ray resection and ray anterpolation operations to obtain incoming rays of level \( i \) from the rays of level \( i+1 \).
First, consider the interpolation and anterpolation operations. As mentioned earlier, the outgoing rays associated with a group describe the time dependent radiation pattern of the source distribution associated with that group. Because the radiation pattern of a source distribution that is spatially bounded by a sphere of radius \( R_s(i) \) and spectrally bandlimited to \( \omega_s \) can be expressed in terms of \( K(i) \) spherical harmonics [98], the interpolation operator increases the sampling rate and zero-pads the excess spherical spectrum introduced. Similarly, anterpolation operations call for the application of a spherical filter with uniform resolution to a set of outgoing or incoming rays. In other words, anterpolation is equivalent to truncating the spherical harmonic content and lowering the sampling rate over the sphere. If implemented as described by Jakob-Chien and Alpert [102], the application of interpolation and anterpolation operators between levels \( i \) and \( i+1 \) can be completed in \( O(K^2(i)\log K(i)) \) operations per time step and per child-parent group pair. This yields approximately \( O(N_s \log K(i)) \) operations per time step at level \( i \), and an overall computational cost of order less than \( O(N_i N_s \log^2 N_s) \).

The nature of the splicing and resection operations can be understood by inspecting Eq. (3.20). With reference to Figure 3.10(a), assume that an outgoing ray of a level \( i+1 \) box, whose center is denoted by \( r_c \), is to be formed. Denoting the centers of the \( N_c \) child boxes associated with this level \( i+1 \) box as \( r_{c,\xi}, \xi = 1, \ldots, N_c \), and the part of the surface \( S \) that lies inside each of these level \( i \) boxes as \( S_{\xi} \), the expression for an outgoing ray of the parent box takes the form

\[
\int_S \delta \left[ t + \hat{k}_{pq} \cdot (r' - r_c)/c \right] q_l^{i+1}(r', t) = \sum_{\xi=1}^{N_c} \delta \left[ t + \hat{k}_{pq} \cdot (r_{c,\xi} - r_c)/c \right] \int_{S_{\xi}} \delta \left[ t + \hat{k}_{pq} \cdot (r' - r_{c,\xi})/c \right] q_l^{i+1}(r', t). \tag{3.27}
\]

In Eq. (3.27), the superscript \( i+1 \) on \( q_l(r', t) \) signifies the fact that the source subsignal is of duration \( T_s(i+1) \), and \( \hat{k}_{pq} \) denotes a ray direction at level \( i+1 \). Since each \( q_l^{i+1}(r, t) \) can be
obtained by splicing two level \( i \) subsignals \( q^i_r (r, t) \) and \( q^{i'}_{l' + 1}(r, t) \) for some \( l' \) as shown in Figure 3.10(b), the outgoing ray of the parent box can be expressed as

\[
\sum_{\zeta = l'}^{l' + 1} \sum_{\xi = 1}^{N_\zeta} \delta \left[ t + \hat{k}_{pq} \cdot (r_{c, \zeta} - r_c) / c \right] \ast \int_{S'_\zeta} dr' \delta \left[ t + \hat{k}_{pq} \cdot (r' - r_{c, \xi}) / c \right] q^i_r (r', t) \cdot q^i_{l' + 1}(r, t) \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 
\]

Figure 3.10: Schematic description of the splicing and resection operations.

Note that in Eq. (3.28), the integral over each \( S'\zeta \) is nothing but an outgoing ray of a child box propagating along direction \( \hat{k}_{pq} \). Hence, Eq. (3.28) clearly indicates that once the outgoing rays of the child boxes are interpolated to level \( i + 1 \) directions \( \hat{k}_{pq} \), they can be time advanced by an amount \( s_{\xi}(\hat{k}_{pq}) = \hat{k}_{pq} \cdot (r_{c, \xi} - r_c) / c \) and spliced together to form the outgoing rays of the parent box. In a completely complementary manner, it can be shown that the incoming rays of a
level \( i \) group can be obtained by resecting the incoming rays of its parent box as depicted in Figure 3.10(b), delaying these rays by an amount \( s_{\hat{g}}(\mathbf{k}_{pq}) \), and anterpolating the resulting rays to level \( i \) directions.

With all requisite tools for a multilevel scheme defined, the fast multilevel evaluation of the sum in Eq. (3.5) proceeds as follows:

1. **Evaluation of the near-field contributions.** As in the two-level scheme, all interactions between source/observer combinations that belong to near-field pairs are accounted for classically. As mentioned earlier, all near-field pairs reside on the finest level. Since each finest level group interacts with only a small number of its immediate neighbors through the near field, the computational complexity associated with the evaluation of the near-field contributions per time step scales as \( O(N_g(1)M_s^2(1)) \). However, since \( N_g(1) \ll N_s \) and \( M_s(1) \) is of \( O(1) \), the total cost for near-field evaluation for all time steps scales as \( O(N_tN_s) \).

2. **Evaluation of the far-field contributions.** These contributions are still evaluated via a three-stage process reminiscent of that described for the two-level scheme. However, the independent generation of outgoing and incoming rays at all levels would result in too high a computational cost. Fortunately, these rays can be efficiently constructed by supplementing the first and the last stages by the ray interpolation, splicing, resection, and anterpolation operations. With these modifications in mind, the three-stage far-field evaluation scheme can be described as follows:

   a) **Construction of outgoing rays.** Outgoing rays for all groups at level \( i \) are constructed every \( M_t(i) \) time steps. At level 1, this is accomplished by convolving the source signatures with \( V^+_n(\mathbf{k}_{pq}, t) \). Higher-level rays are constructed from those at the previous
level through interpolation and splicing. Note that, since the fundamental subsignal duration at level \( i + 1 \) is twice that at level \( i \) (Figure 3.9(a)), the levels must be traversed starting from level 1. This guarantees that level \( i \) rays are constructed before they are needed by the ray splicing operator to obtain the rays at level \( i + 1 \). As mentioned above, the total cost of constructing outgoing rays for all levels using interpolation and splicing scales as \( O(N_t N_s \log^2 N_s) \).

b) **Translation.** As in the two-level algorithm, rays are translated between the far-field pairs \((\gamma, \gamma')\) at level \( i \) every \( M_{t,\gamma\gamma'}(i) \) time steps by concatenating rays of fundamental duration \( T_\gamma(i) \) formed in Step 2a. Note that a different \( \tilde{T}(\theta', \Omega) \) table has to be constructed for each level since the number of harmonics \( K(i) \) required in the computation of the translation functions varies across levels. The translation of rays for all level \( i \) far-field pairs can be accomplished in

\[
O(N_g(i)K^2(i)(N_t/M_{t,\gamma\gamma'}(i))M_{t,\gamma\gamma'}(i)\log M_{t,\gamma\gamma'}(i)) = O(N_s N_t \log M_s(i))
\]

operations. When summed over all levels, this yields a complexity of

\[
O(N_t N_s \sum_{i=1}^{\log N_s} \log M_s(i)) = O(N_t N_s \log^2 N_s)
\]

for translation.

c) **Projection of incoming rays onto observers.** Starting with level \( N_t - 1 \), incoming rays for each group are resected and anterpolated from the incoming rays at the higher level every \( M_t(i) \) time steps. The fields at the observer locations are constructed by convolving incoming rays at level 1 with \( \tilde{V}_n^{-}\left(\hat{k}_{pq}, t\right) \) and performing the spherical integration. This step, which is conceptually the transpose of forming the outgoing rays, can also be accomplished in \( O(N_t N_s \log^2 N_s) \) operations.

Considering all the steps of the multilevel algorithm, it is seen that a scattering analysis using a multilevel PWTD enhanced MOT scheme can be completed in \( O(N_t N_s \log^2 N_s) \) operations.
3.5 The Windowed Plane-Wave Time-Domain Algorithm

The previous section has shown that two-level and multilevel PWTD-enhanced algorithms offer a considerable reduction in computational complexity over traditional MOT schemes. As with frequency domain fast multipole methods, the computational complexity of the proposed schemes can be further reduced by using windowed translation functions [58, 90]. This additional cost reduction hinges on the observation that, if the time span of the translation functions can be shortened, the number of rays that need to be translated for the evaluation of the observed field shrinks to a constant independent of the group size for sufficiently remote source and observation groups. Shorter translation functions can be constructed either by smoothly tapering off, or windowing, the series in Eq. (3.19) for \( k > K \) instead of truncating it at \( K \) or by using sampled field representations that rely on the bandlimitedness of the radiated fields as will be done in the present section.

This section introduces windowed diagonal time domain translation operators that permit the rapid evaluation of transient fields produced by surface-bound source distributions. It will be shown that the computational complexities associated with the solution of large scale surface scattering problems using the proposed two-level and multilevel windowed PWTD algorithms based on these operators scale as \( O(N_t N_s^{4/3} \log N_s) \) and \( O(N_t N_s \log N_s) \), respectively. Computer codes based on these PWTD algorithms are expected to outperform classical MOT and FDTD codes for sufficiently large surface scatterers.

3.5.1 Windowed plane wave decomposition

The immediate goal of this section is to develop a plane wave representation of the field \( u(r, t) \) radiated by the source distribution \( q(r, t) \) that is more economical than the one presented in the previous section. Again, it is assumed that \( u(r, t) \) is related to \( q(r, t) \) as in Eq. (3.2), and
that both the source and field are sectioned as in Eqs. (3.7) and (3.8). To arrive at a windowed plane wave representation of \( u_l(r, t) \), consider the field \( \tilde{u}_l(r, t) \) defined by

\[
\tilde{u}_l(r, t) = \frac{-\partial_t \theta_{\text{int}}}{8\pi^2 c} \int_0^{\theta_{\text{int}}} d\theta \sin \theta \int_0^{2\pi} d\phi \int_{S} d\mathbf{r}' \delta \left[ t - \hat{k} \cdot (r - r')/c \right] \cdot q_l(r', t).
\]  

(3.29)

Note that Eq. (3.29) is different from Eq. (3.9) in that the integration is over a cap of the unit sphere for which \( \theta \leq \theta_{\text{int}} \) as illustrated in Figure 3.11(a). Also note that for convenience the z-axis is chosen such that it is aligned with the \( \mathbf{R}_c \) vector, i.e., \( \mathbf{R}_c = R_c \hat{z} \). To gain insight into the relationship between \( \tilde{u}_l(r, t) \) and \( u_l(r, t) \), the integral over the spherical cap is evaluated by transforming the integration variables \( (\theta, \phi) \) to a new set of angular coordinates \( (\theta', \phi') \) which are defined with respect to the axis aligned with the vector \( \mathbf{R} \) as shown in Figure 3.11(a). In this new coordinate system, the upper limit on \( \theta' \), \( \theta_{\text{int}} \), depends on \( \phi' \), \( r \), and \( r' \), and

\[
\tilde{u}_l(r, t) = -\frac{\partial_t}{8\pi^2 c} \int_{S} d\mathbf{r}' \int_0^{2\pi} d\phi' \int_0^{\theta_{\text{int}}(\phi', r, r')} d\theta' \sin \theta' \delta \left( t - \hat{k}' \cdot \mathbf{R}'/c \right) \cdot q_l(r', t)
\]  

(3.30)

where \( \hat{k}' = \hat{z}' \sin \theta' \cos \phi' + \hat{y}' \sin \theta' \sin \phi' + \hat{x}' \cos \theta' \), and \( \mathbf{R}' = \hat{z}' \mathbf{R} \). In deriving the limits on the elevation integral in (3.30), it was tacitly assumed that

\[
\theta_{\text{int}} > \cos^{-1}(\hat{z} \cdot \mathbf{R}/R).
\]  

(3.31)

Using \( \hat{k}' \cdot \mathbf{R}' = R \cos \theta' \) and setting \( \tau = R \cos \theta'/c \) in (3.30) yields

\[
\tilde{u}_l(r, t) = \int_{S} d\mathbf{r}' \int_0^{2\pi} d\phi' \frac{R/c}{-(R/c) \cos \theta_{\text{int}}(\phi', r, r')} \int_0^{\theta_{\text{int}}(\phi', r, r')} d\tau \frac{\partial_t \delta(t - \tau)}{8\pi^2 R} \cdot q_l(r', t)
\]

\[
= \int_{S} d\mathbf{r}' \frac{\delta(t - R/c)}{4\pi R} \cdot q_l(r', t) \bigg[ -\int_{S} d\mathbf{r}' \frac{\delta(t + R \cos \theta_{\text{int}}(\phi', r, r')/c)}{4\pi R} \cdot q_l(r', t) \bigg]
\]

(3.32)

\[
= u_l(r, t) - \int_{S} d\mathbf{r}' \frac{\delta(t + R \cos \theta_{\text{int}}(\phi', r, r')/c)}{4\pi R} \cdot q_l(r', t).
\]  

83
In Eq. (3.32), the first term on the right-hand side corresponds to the true observer field $u_i(r, t)$. Note that, as in Eq. (3.10), there is again a ghost signal represented by the second term. However, the ghost signal in Eq. (3.32) is spread out and spans a longer duration than the one in Eq. (3.10). The above derivation closely follows that of Heyman [6], who generalizes the results of Tygel and Hubral [2]. Equations (3.29) and (3.32) imply that, if the ghost signal can somehow be removed from $\tilde{u}_i(r, t)$, the true observer field can be constructed as a superposition of plane waves using techniques that are akin to those underlying the frequency-domain fast multipole method. In what follows, a scheme is derived that permits to time gate $\tilde{u}_i(r, t)$ in order to retain only the true observer field.

![Diagrams](image-url)

Figure 3.11: (a) Definition of angles in primed coordinate system. (b) Relevant dimensions for ray translation.

It can be verified that Eq. (3.31) holds for arbitrary source and observer locations $r'$ and $r$ chosen within the source and observation spheres, respectively, provided that
\[ \theta_{\text{int}} > \sin^{-1}(2R_s/R_c). \]  

From Eq. (3.32) it follows that the ghost signal present in \( \tilde{u}_l(r, t) \) vanishes after

\[ t_l^\text{ghost} = \frac{R}{c} \cos \theta_{\text{min}}' + (l+1)T_s \]

\[ < \left( \frac{R_c \cos \theta_{\text{int}} + 2R_s}{c} + (l+1)T_s, \right. \]

where \( \theta_{\text{min}}' = \min[\theta_{\text{int}}'(\phi', \mathbf{r}, \mathbf{r}')], \) and the upper bound follows from geometrical considerations (see Figure 3.11(b)). The fields in the observation sphere coincide with the true fields after the ghost signal has vanished. Also, the true field does not reach the observation sphere before

\[ t_l^\text{trans} = \left( \frac{R_c - 2R_s}{c} \right) + lT_s. \]

Therefore, provided that \( t_l^\text{trans} > t_l^\text{ghost} \), all ghost fields in the observation sphere cease to exist before the true signal arrives. In addition, assuming that \( q_l(\mathbf{r}, t) = 0 \) outside the interval \( lT_s \leq t < (l+1)T_s \), if \( t_l^\text{trans} > (l+1)T_s \), all source activity related to the \( l^{th} \) time interval ends before the true signal reaches any observer. In summary,

\[ t_l^\text{trans} \geq t_l^\text{ghost} \Rightarrow u_l(\mathbf{r}, t) = \begin{cases} 
0 & t < t_l^\text{trans} \\
\tilde{u}_l(\mathbf{r}, t) & t \geq t_l^\text{trans}
\end{cases}, \]

\[ t_l^\text{trans} \geq (l+1)T_s \Rightarrow q_l(\mathbf{r}, t) = 0, \ t \geq t_l^\text{trans}. \]

The above two conditions can be restated, using (3.34) and (3.35), as

\[ \frac{cT_s}{R_s} \leq \frac{R_c}{R_s} - 2 \]  

and

\[ \frac{cT_s}{R_s} \leq \frac{R_c}{R_s}(1 - \cos \theta_{\text{int}}) - 4. \]
It can be shown that constraint (3.33) is automatically satisfied provided that (3.39) holds for any $T_s \geq 0$.

The above two constraints are key to the development of the windowed PWTD algorithm. Eq. (3.36) implies that if, for a given source and observation sphere pair (i.e., for a given $R_c/R_s$), a $cT_s/R_s$ and a $\theta_{int}$ that satisfy both (3.38) and (3.39) are selected, then the field $u(r,t)$ can be reconstructed as a superposition of time-gated $\tilde{u}_l(r,t)$. The contribution of each of the time-gated $\tilde{u}_l(r,t)$ to the observed field can be obtained by translating the SST of the source distribution $q_l(r,t)$ at $t = t_l^{\text{trans}}$. It is easily recognized that the SST of $q_l(r,t)$ corresponds to "outgoing" rays, leaving the source sphere; similarly, it will be shown in the next subsection that after translation (i.e., after $t = t_l^{\text{trans}}$), $u_l(r,t)$ can be described in terms of "incoming" rays impinging upon the observation sphere. Condition (3.38) ensures that this SST can be completely constructed prior to the translation time, enabling the PWTD algorithm to be incorporated into any time marching scheme.

In practice, provided that a $cT_s/R_s$ that satisfies (3.38) is chosen for a given $R_c/R_s$, $\theta_{int}$ is computed from (3.39) by enforcing the equality. This procedure minimizes $\theta_{int}$ and hence will minimize the computational cost associated with the numerical procedure for evaluating $\tilde{u}_l(r,t)$, as described in Section 3.6. For this choice of $cT_s/R_s$ and $\theta_{int}$, it follows from (3.34)-(3.36) that at $t = t_l^{\text{trans}}$, the ray traveling along $\theta = 0$ is about to enter the observation sphere, while rays traveling along directions $\theta = \theta_{int}$ have all exited the sphere (Figure 3.11(b)). At $t = t_l^{\text{trans}}$, rays traveling at intermediate angles partially overlap with the observation sphere, but add up to a null field in its interior.

The implications of inequalities (3.38) and (3.39) are further illustrated in Figure 3.12. For a given $R_c/R_s$, combinations of $\theta_{int}$ and $cT_s/R_s$ that satisfy both (3.38) and (3.39) lie to the lower right of the intersection of the curves obtained by enforcing the equalities in (3.38) and
(3.39). For example, while the point \((\theta_{\text{int}}, cT_s/R_s) = (120^\circ,15)\) permits a ghost-free solution for \(R_c/R_s = 20\), this same combination does not permit a ghost-free solution for \(R_c/R_s = 10\). Note that as the two spheres approach each other, the region that satisfies both conditions collapses to the point \((\theta_{\text{int}}, cT_s/R_s) = (180^\circ,0)\).

![Graphical representation of the constraints (3.38) and (3.39) for different values of \(R_c/R_s\).](image)

Figure 3.12: Graphical representation of the constraints (3.38) and (3.39) for different values of \(R_c/R_s\).

### 3.5.2 Implementation using sampled field representations

Equation (3.29), together with constraints (3.38) and (3.39), is the basis for formulating the windowed PWTD algorithm for sampled field representations. In practice, \(u(r,t)\) is evaluated by assuming that the source distribution \(q(r,t)\) is temporally bandlimited, i.e., the temporal spectrum of \(q(r,t)\) vanishes for \(\omega > \omega_{\text{max}}\). Hence, \(q(r,t)\) can be sampled and locally interpolated using temporally bandlimited and approximately time limited functions as

\[
q(r,t) = \sum_{k=1}^{N_t} q(r,k\Delta_t) \psi_k(t),
\]

(3.40)
where $\Delta_t$ is the time step size and $\psi_k(t)$ is a bandlimited interpolant [103]. Many choices for the interpolation function exist; however, a near optimal one—a variant of the approximate prolate spheroidal (APS) functions introduced by Knab [104]—is given by

$$
\psi_k(t) = \frac{\omega_+ \sin(\omega_+(t - k\Delta_t))}{\omega_s \omega_+(t - k\Delta_t)} \cdot \frac{\sinh\left(\omega_- p_t \Delta_t \sqrt{1 - \left[\frac{(t - k\Delta_t)}{p_t\Delta_t}\right]^2}\right)}{\sinh(\omega_- p_t \Delta_t) \sqrt{1 - \left[\frac{(t - k\Delta_t)}{p_t\Delta_t}\right]^2}},
$$

(3.41)

where

$$
\omega_s = \pi/\Delta_t = \chi_o \omega_{\text{max}}.
$$

(3.42)

In the above, $\chi_o > 1$ is the oversampling ratio, $\omega_\pm = (\omega_s \pm \omega_{\text{max}})/2$, and $p_t$ is an integer that defines the approximate duration of the interpolation function. In practice, a truncated version of $\psi_k(t)$, obtained by setting $\psi_k(t) = 0$ for $|t - k\Delta_t| > p_t\Delta_t$, is used. The relative interpolation error $\varepsilon_t$ introduced by this truncation can be shown to be bounded as [104]

$$
|\varepsilon_t| \leq \frac{1}{\sinh(\omega_- p_t \Delta_t)},
$$

(3.43)

which decreases exponentially fast with increasing $p_t$. Hence, Eq. (3.40) permits local interpolation in terms of $2p_t$ samples.

From Eq. (3.40), it follows that the source signal $q(r,t)$ can be broken up into subsignals $q_l(r,t)$, as in Eq. (3.7), given by

$$
q_l(r,t) = \sum_{k=lM_t}^{(l+1)M_t-1} q(r, k\Delta_t) \psi_k(t).
$$

(3.44)

Each subsignal $q_l(r,t)$ is defined in terms of $M_t$ samples of the signal $q(r,t)$ but spans $M'_t = M_t + 2p_t$ time steps. In other words, whereas each subsignal $q_l(r,t)$ is formed from samples of $q(r,t)$ in an interval of length $T_s = M_t \Delta_t$, the duration of each $q_l(r,t)$ is $T'_s = M'_t \Delta_t$, 88
and adjacent subsignals overlap by $2p_t$ samples (Figure 3.2(b)). Obviously, the total number of
time samples equals $N_t = LM_t$. Since the interpolation function $\psi_k(t)$ is bandlimited to $\omega_s$, so
is each subsignal.

Proceeding with the derivation of a closed-form translation operator, note that

$$\int_S \delta(t - \hat{\mathbf{k}} \cdot \mathbf{R}/c) \delta(t - \hat{\mathbf{k}} \cdot \mathbf{r}_o/c) \delta(t - \hat{\mathbf{k}} \cdot \mathbf{R}_c/c) \ast q_{l\text{out}}(\hat{\mathbf{k}}, t),$$

where $\mathbf{r}_o = \mathbf{r} - \mathbf{r}_o$ and $q_{l\text{out}}(\hat{\mathbf{k}}, t)$ the SST of $q_l(\mathbf{r}, t)$ as given in Eq. (3.13). Using (3.45) in
(3.29) and letting $\mathbf{r}_s$ denote $(\mathbf{r}' - \mathbf{r}_s)$ yields

$$\bar{u}_{l}(\mathbf{r}, t) = -\frac{\delta t}{8\pi^2 c} \int_0^{2\pi} \int_0^\theta \delta(\mathbf{t} - \hat{\mathbf{k}} \cdot \mathbf{r}_s/c) \delta(\mathbf{t} - \hat{\mathbf{k}} \cdot \mathbf{R}_c/c)$$

$$\ast \int_S \delta(\hat{\mathbf{k}} \cdot \mathbf{r}_o/c) \ast q_{l}(\mathbf{r}', t). \quad (3.46)$$

As mentioned earlier, the SST of the source distribution represented by the last convolution in
Eq. (3.46) can be interpreted as outgoing rays leaving the source sphere. It is seen that for a
point source, the SST imposes a direction dependent shift on $q_l(\mathbf{r}, t)$ by an amount of $\hat{\mathbf{k}} \cdot \mathbf{r}_s/c$
which can be incorporated into Eq. (3.40). Note that the leftmost convolution in Eq. (3.46) is the
same as the SST except that the direction dependency is reversed. Therefore, $\bar{u}_{l}(\mathbf{r}, t)$ can be
interpreted as a superposition of incoming rays projected onto the observers. However, the use
of the interpolation function defined in Eq. (3.41) implies that knowledge of the field at the edge
of the observation sphere requires knowledge of samples from incoming rays that reside $p_t$
samples exterior to the sphere in all directions. Therefore, when working with sampled field
representations, constraints (3.38) and (3.39) should be satisfied in terms of $T'_s$ and
$R'_s = R_s + p_t c \Delta t$, instead of $T_s$ and $R_s$. Translation times should also be computed in terms of
the primed quantities.
To efficiently evaluate $\tilde{u}_l(r, t)$, define

$$
\hat{g}_l(\hat{k}, t) = \delta(t - \hat{k} \cdot \hat{r}_o/c) \ast \int_{S} d\hat{r}' \delta(t + \hat{k} \cdot \hat{r}_s/c) \ast q_l(\hat{r}', t) = \int_{S} d\hat{r}' \delta(t - \hat{k} \cdot (\hat{r}_o - \hat{r}_s)/c) \ast q_l(\hat{r}', t).
$$

(3.47)

The function $\hat{g}_l(\hat{k}, t)$ can be interpreted as the time dependent radiation pattern of a source distribution residing in a sphere of radius $2R'_s$. Therefore, $\hat{g}_l(\hat{k}, t)$ is spatially quasi-bandlimited and can be reconstructed to arbitrary precision, provided it is sampled densely enough over the sphere, using the expansion [99]

$$
\hat{g}_l(\hat{k}, t) = \sum_{n=0}^{M'} \sum_{m=-M_n}^{M_n} \hat{g}_l(\hat{k}_{nm}, t) \Omega_{nm}(\hat{k}),
$$

(3.48)

where the functions $\Omega_{nm}(\hat{k})$ represent bandlimited spherical interpolation functions. As with the temporal interpolants, many different choices for the $\Omega_{nm}(\hat{k})$ exist. One near optimal choice is a combination of the Dirichlet kernel and the cylindrical APS function introduced by Bucci et al. [99], for which

$$
\hat{k}_{nm} = \hat{x} \sin \theta_n \cos \phi_{nm} + \hat{y} \sin \theta_n \sin \phi_{nm} + \hat{z} \cos \theta_n,
$$

(3.49)

$$
\phi_{nm} = m2\pi/(2M_n + 1),
$$

(3.50)

$$
\theta_n = n2\pi/(2M' + 1),
$$

(3.51)

$$
M' = \text{Int}(\chi_2 M),
$$

(3.52)

$$
M = \text{Int}(2\chi_1 \omega_s R'_s/c) + 1,
$$

(3.53)

$$
M_n = \text{Int}\left(\frac{2(\sin \theta_n + (\chi_1 - 1) \sin^{1/3} \theta_n) \omega_s R'_s/c}{c} + 1\right),
$$

(3.54)
\[\Omega_{nm}(\hat{k}) = \begin{cases} S_0(\theta) & n = 0 \\ S_n(\theta)D_{M_n}(\phi - \phi_{nm}) + S_n(-\theta)D_{M_n}(\phi + \pi - \phi_{nm}) & n \neq 0 \end{cases} \tag{3.55}\]

\[D_{M_n}(\phi) = \frac{\sin[(2M_n + 1)\phi/2]}{(2M_n + 1)\sin(\phi/2)} \tag{3.56}\]

\[S_n(\theta) = \frac{R_N(\theta - \theta_n, p_s\Delta\theta)}{R_N(0, p_s\Delta\theta)} D_{M'}(\theta - \theta_n) \tag{3.57}\]

\[\Delta\theta = 2\pi/(2M' + 1) \tag{3.58}\]

\[R_N(\theta, p_s\Delta\theta) = \frac{\sinh[(2N + 1)\sinh^{-1}(\sqrt{\sin^2(p_s\Delta\theta/2) - \sin^2(\theta/2)})]}{\sqrt{\sin^2(p_s\Delta\theta/2) - \sin^2(\theta/2)}} \tag{3.59}\]

In the above, \(\chi_1 > 1\) is the excess bandwidth factor, \(\chi_2 > 1\) is the oversampling ratio in elevation, \(p_s\) is an integer that defines the approximate angular extent of \(S_n(\theta)\), and \(N = M' - M\). As with the temporal interpolation functions, \(S_n(\theta)\) can also be truncated for \(|\theta - \theta_n| > p_s\Delta\theta\) yielding a relative interpolation error \(\varepsilon_s\) bounded by

\[|\varepsilon_s| \leq \frac{1}{\sinh[\pi p_s(1 - 1/\chi_2)]} \tag{3.60}\]

This error also decreases exponentially fast with increasing \(p_s\). Hence, Eq. (3.48) permits local interpolation in elevation in terms of \(2p_s\) samples.

Substituting expansion (3.48) in (3.46), rearranging the terms, and interchanging the order of summations and integrations yield

\[\tilde{u}_l(r_0, t) = \sum_{n=0}^{M'} \sum_{m=-M_n}^{M_n} \delta \left(t - \hat{k}_{nm} \cdot \hat{r}_0/c\right) \ast T_{nm}(R_c, t) \ast \int_{S} dr' \delta \left(t + \hat{k}_{nm} \cdot \hat{r}_s/c\right) \ast q_l(r', t), \tag{3.61}\]

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where
\[
\mathcal{T}_{nm}(\mathbf{R}_c, t) = -\frac{\partial}{\partial t} \int_0^{2\pi} d\phi \int_0^{\theta_{int}} d\theta \sin \theta \quad \Omega_{nm}(\hat{\mathbf{k}}) \delta\left(t - \hat{\mathbf{k}} \cdot \mathbf{R}_c/c\right)
\]  
\[ (3.62) \]

is the translation function. Since \( \hat{\mathbf{k}} \cdot \mathbf{R}_c = R_c \cos \theta \), the integral in Eq. (3.62) can be evaluated in closed form and the translation function can be succinctly expressed as
\[
\mathcal{T}_{nm}(\mathbf{R}_c, t) = -\frac{\partial}{\partial t} \frac{1}{4\pi R_c (2M_n + 1)} \Psi_n \left( \cos^{-1} \frac{ct}{R_c} \right) \quad \text{for} \quad \frac{R_c}{c} \cos \theta_{int} \leq t \leq \frac{R_c}{c}, \quad (3.63)
\]

where
\[
\Psi_n(\theta) = \begin{cases} 
S_0(\theta) & n = 0 \\
S_n(\theta) + S_n(-\theta) & n \neq 0
\end{cases} \quad (3.64)
\]

Another useful expression for \( \mathcal{T}_{nm}(\mathbf{R}_c, t) \) results upon expanding the spatially bandlimited and even functions \( \Psi_n(\theta) \) in a cosine series as
\[
\Psi_n(\theta) = \sum_{k=0}^{M' + N} a_{n,k} \cos(k\theta). \quad (3.65)
\]

Substituting (3.65) into (3.63), and using the relation \( T_k(x) = \cos(k \cos^{-1} x) \), where \( T_k(x) \) is the \( k^{th} \)-order Chebyshev polynomial, yields
\[
\mathcal{T}_{nm}(\mathbf{R}_c, t) = -\frac{\partial}{\partial t} \frac{1}{4\pi R_c (2M_n + 1)} \sum_{k=0}^{M' + N} a_{n,k} T_k \left( \frac{ct}{R_c} \right) \quad \text{for} \quad \frac{R_c}{c} \cos \theta_{int} \leq t \leq \frac{R_c}{c}. \quad (3.66)
\]

Equation (3.66) shows that the translation function can be expressed as a finite-order polynomial. The sum in Eq. (3.66) can be efficiently evaluated using Clenshaw's recurrence algorithm [105].

To elucidate the properties of the translation function, the functions \( \Psi_n(\theta) \) are plotted with respect to \( \theta \) for \( n = 0, \ldots, 10 \), \( M = 4 \), \( M' = 10 \), and \( p_z = 3 \) in Figure 3.13(a). The
corresponding time signals \( \Psi_n(\cos^{-1}(ct/R_c)) \) are shown in Figure 3.13(b) as a function of the time parameter \( \tau = ct/R_c \). Clearly, the duration of the translation function is \((1 - \cos \theta_{\text{int}})R_c/c\) . If \( \theta_{\text{int}} \) is chosen as outlined in the concluding paragraphs of Subsection 3.5.1, i.e., by enforcing the equality in (3.39) for a \( T_s'/cR_s' \) that satisfies constraint (3.38), this duration equals \( T_s' + 4R_s'/c \). However, a truncated version of \( \Psi_n(\theta) \) may be used because \( S_n(\theta) \) is vanishingly small for \(|\theta - \theta_n| > p_s\Delta \theta \). As can be seen in Figure 3.13, the translation function associated with the directions for which \( \theta_n > \theta_{\text{int}} + p_s\Delta \theta \) vanishes as \( \Psi_n(\theta) \equiv 0 \) for \( 0 < \theta < \theta_{\text{int}} \). The \( \Psi_n(\theta) \) associated with these directions \((n = 7, \ldots, 10)\) are plotted with dash-dotted lines in Figure 3.13. For other directions, the nonvanishing portion of \( \Psi_n(\theta) \) in the interval \( 0 < \theta < \theta_{\text{int}} \) contributes to the translation function and is plotted with a solid line in Figure 3.13. Note, however, that for these directions, the duration of the translation function may become much shorter than \( T_s' + 4R_s'/c \). Also, if \( T_s' \) is fixed, constraint (3.39) dictates that, as the spheres move further apart, the number of contributing directions decreases.

![Diagram](image)

Figure 3.13: (a) Interpolation functions \( \Psi_n(\theta) \) as a function of \( \theta \). (b) Time signals \( \Psi_n(\cos^{-1}(ct/R_c)) \) as a function of the time parameter \( \tau = ct/R_c \).
The above analysis can easily be extended to source and observation spheres for which \( R_c \) is not aligned with the \( z \)-axis. One approach is to use interpolation functions that are windowed in both elevation and azimuth instead of the above introduced interpolants that are windowed solely in elevation. Alternatively, instead of relying on the bandlimited nature of the far-field interpolation functions to bandlimit the translation operator as in the above derivation, the translation operator can be explicitly bandlimited and the integration over the sphere performed using an exact quadrature formula. This procedure, which was demonstrated in Section 3.3.2, leads to a translation function for which the angular and temporal dependences can be expressed in terms of Legendre polynomials as in Eq. (3.19). This derivation is akin to the traditional construction of windowed translation operators for frequency domain fast multipole methods [52, 58, 90].

### 3.6 Implementation of the Windowed PWTD Enhanced MOT Schemes

This section describes the practical implementation of the windowed PWTD algorithm and studies the computational complexity of MOT schemes supplemented with this algorithm in two-level and multilevel settings. Section 3.6.1 outlines the use of the windowed PWTD for evaluating fields at multiple observers due to multiple sources residing in geometrically separate observation and source spheres. This section also describes a series of numerical tests that were conducted to validate the algorithm. Subsections 3.6.2 and 3.6.3 comment on the use of the windowed PWTD algorithm for scattering computations in a two-level and a multilevel framework, respectively.

#### 3.6.1 Sphere-to-sphere translation

This section outlines a sequence of operations leading to a successful implementation of the windowed PWTD algorithm for computing observer fields. Here, we expand our viewpoint
and assume that \( q(r, t) \) consists of \( M_s \) point sources located at \( r^j_s \), \( j = 1, \ldots, M_s \), distributed throughout the source sphere and characterized by temporal signatures \( f^j(t) \), i.e.,

\[
q(r, t) = \sum_{j=1}^{M_s} f^j(t) \delta(r - r^j_s).
\]  

(3.67)

It is assumed that the spectra of all \( f^j(t) \) vanish for \( \omega > \omega_{\text{max}} \). The field due to \( q(r, t) \) is to be evaluated at \( M_o \) observers located at \( r^i_o \), \( i = 1, \ldots, M_o \), distributed throughout the observation sphere. To simplify the notation, the positions of the source and observation points relative to their respective sphere centers are denoted by \( \tilde{r}^j_s = r^j_s - r_s \) and \( \tilde{r}^i_o = r^i_o - r_o \).

The field at the \( i^{\text{th}} \) observer is given by

\[
u_i(r^i_o, t) = \frac{M_s}{4\pi} \frac{\sum_{j=1}^{M_s} f^j(t - |r^i_o - r^j_s|/c)}{|r^i_o - r^j_s|}.\]  

(3.68)

The evaluation of (3.68) at all \( M_o \) observers for all \( N_t \) time steps is a computationally expensive task as its cost scales as \( O(N_t M_s M_o) \), which is of \( O(N_t M_s^2) \) if \( M_o \propto M_s \). This is due to the fact that in a direct evaluation of Eq. (3.68) for all observers, one has to aggregate the effects of all sources for all time steps.

Alternatively, the fields at the \( M_o \) observers can be evaluated using Eq. (3.61), which, for the source density expressed by (3.67), takes the form

\[
\tilde{u}_l(r^i_o, t) = \sum_{n=0}^{M_n'} \sum_{m=-M_n}^{M_n} \delta \left( t - \tilde{k}_{nm} \cdot \tilde{r}^i_o / c \right) * T_{nm}(R_c, t) * \sum_{j=1}^{M_s} \delta \left( t + \tilde{k}_{nm} \cdot \tilde{r}^j_s / c \right) * f^j_l(t),
\]  

(3.69)

where it is assumed that each source signal \( f^j(t) \) is broken up into \( L \) subsignals \( f^j_l(t) \), \( j = 1, \ldots, M_s \). Equation (3.69) is the crux of the PWTD algorithm and indicates that the fields within the observation sphere can be constructed via a three-step process consisting of source
aggregation, ray translation, and ray disaggregation. The aggregation step, which is represented by the innermost summation in Eq. (3.69), maps the source subsignals onto a set of time-dependent plane waves—henceforth termed subrays—propagating along the $\mathbf{k}_{nm}$ vectors. The translation step is carried out by convolving these subrays with the translation functions given in Eq. (3.66). The disaggregation process can be viewed as the reverse of the aggregation process and maps a set of incoming rays onto observer locations.

To implement this algorithm for a given $R_s'$ and $R_c$, a $T_s'$ is selected which (i) satisfies constraint (3.38) and for which (ii) $R_s'/(cT_s')$ is of $O(1)$. If it is impossible to satisfy both of these conditions, the fields in the observer sphere should be computed using classical procedures, as the PWTD becomes less efficient. Next, a $\theta_{int}$ is computed by enforcing the equality in constraint (3.39). The following three operations are then performed for all $L$ time intervals:

1. Compute the sampled SST of the source distribution for all ray directions (i.e., perform the right most convolution and carry out the innermost summation in Eq. (3.69)).

2. At $t = t_{trans}^l$, convolve each subray with the translation function on a direction-by-direction basis, and add the resulting subrays onto incoming rays which propagate through the observation sphere (i.e., perform the middle convolution in Eq. (3.69)).

While in certain cases it is advantageous to perform this convolution directly in the time domain, it is assumed here that the convolution is performed using an FFT. However, care should be exercised as the translation function is not bandlimited and cannot be sampled without aliasing. On the other hand, since each subray is bandlimited, so is the result of the convolution. In practice, the Fourier transform of the translation function is evaluated analytically at the frequency points required by the FFT. This is efficiently accomplished by
locally expanding the translation function in terms of a small set of orthogonal polynomials whose Fourier transforms are well defined.

Note that this operation translates each subray onto an incoming ray that propagates in the same direction, analogous to diagonal frequency domain fast multipole translation operators.

3. Evaluate the fields at the observers as the incoming rays travel across the observation sphere (i.e., perform the left most convolution and summations in Eq. (3.69)).

Note that each subray can be at most \((2R_s^c/c + T_s^c)/\Delta_t\) time steps long. Furthermore, as discussed in Subsection 3.5.2, the translation function associated with each ray direction is \((4R_s^c/c + T_s^c)/\Delta_t\) time steps long. By virtue of the choice \(cT_s^c \propto R_s^c\), both the subray duration and the translation function length scale as \(O(M_s^t)\). From Eq. (3.48), it is seen that the number of ray directions \(D_s\) equals \(\sum_{n=0}^{M'_s} (2M_n + 1)\). Using Eqs. (3.52)-(3.54), it can be shown that \(D_s\) is proportional to the surface area of each sphere, i.e., \(D_s \propto (R_s^c/(c\Delta_t))^2\).

Since the aggregation step maps \(M_s\) source subsignals onto \(D_s\) subrays, its computational cost scales as \(O(M_t^s M_s D_s)\). The dominant cost in the translation step is due to the convolution and scales as \(O(M_t^s \log M_t^s)\) if evaluated using an FFT. This operation is performed for all \(D_s\) directions, yielding a computational complexity of \(O(D_s M_t^s \log M_t^s)\). The disaggregation step has the same complexity as the aggregation step.

For a surface scatterer, the number of sources or observers in a sphere is proportional to the surface area of the sphere, i.e., \(M_s \propto (R_s^c/(c\Delta_t))^2\). This implies that \(M_t^s \propto \sqrt{M_s}\), and that \(D_s \propto M_s\). It may be verified that the costs of the aggregation and disaggregation processes dominate that of the translation process and that the cost of evaluating the observed fields for one subsignal using PWTD algorithm scales as \(O(M_t^s M_s^2)\). Hence, the cost of evaluating the fields due to all \(L\) subsignals scales as \(O(N_t M_s^2)\). This cost is no less than that of the classical
algorithm. Nonetheless, the PWTD scheme permits the reuse of SST information, which results in a reduction of the computational complexity when applied in an integral-equation setting.

However, as noted previously, not all outgoing rays contribute to the observed fields if use is made of the windowed character of the translation operator. In fact, as is evident by Figure 3.12, the number of subrays that need to be translated shrinks to a constant as the ratio of the distance between the spheres and the sphere radii increases. Under these circumstances, $D_s$ can be omitted in the above complexity estimates; hence, the computational complexity of computing the fields associated with one time interval scales as $O(M'_t M_s)$, and that of all $L$ subsignals combined scales as $O(N_t M_s)$.

A series of numerical experiments was conducted to validate and examine accuracy versus efficiency tradeoffs in the windowed PWTD algorithm. To this end, the fields due to a set of point sources are calculated using the windowed PWTD scheme and compared to the exact fields. In what follows, all source functions are generated using the Gaussian pulse

$$g(t) = e^{-(t-7.75\sigma)^2/2\sigma^2}.$$  \hspace{1cm} (3.70)

The variance $\sigma$ is fixed at 2.12 ns, which yields a pulse with a duration (full-width between half maximum) of 5 ns. For all practical purposes, this pulse can be assumed to be bandlimited to $\omega_{\text{max}} = 600\pi \times 10^6$ rad/s. The source and observation spheres are of radius $R_s = 1$ m and are separated by $R_c = 20$ m. The time step size $\Delta_t$ is fixed at 0.5 ns and the wave speed of the medium is chosen to be $c = 3 \times 10^8$ m/s. For a sampled field representation, this choice of parameters yields $t_l^{\text{trans}} = 120\Delta_t + IT_s - 2p_l\Delta_t$.

As a first test, the fields due to two sources located at s1 and s2, as shown in the inset of Figure 3.14(a), are evaluated at two observer locations, o1 and o2. The time signatures of the sources are $f^J(t) = g(t) + 0.5g(t - 3 \times 10^{-8})$, $J = 1,2$. In accordance with (3.38), $T_s$ is chosen to

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be $6\Delta_t$, which results in $\theta_{int} = 54.9^\circ$ by the equality in (3.39). As can be seen in Figure 3.14(a), the PWTD results are in excellent agreement with the exact fields. The fields due to the first pulse, $g(t)$, are observed for $t < 98$ ns, and those due to the second pulse, $0.5g(t-3 \times 10^{-8})$, appear later. Figure 3.14(b) shows the observed fields obtained by translating the subrays at $t = 30\Delta_t + 1T_s - 2p_i\Delta_t < t^{\text{trans}}$. As expected, early translation of subrays has produced ghost signals. The ghosts due to $g(t)$ are seen for $t < 67$ ns and those due to $0.5g(t-3 \times 10^{-8})$ corrupt the observed fields in the interval $75$ ns $< t < 98$ ns.

Figure 3.14: Observer responses at o1 and o2 due to sources at s1 and s2 for (a) correct $t^{\text{trans}}$, (b) small $t^{\text{trans}}$.

Next, several tests were conducted to check the accuracy of the PWTD algorithm. For this purpose, six sources with time signatures $f^j(t) = g(t)$, $j = 1, \ldots, 6$, were distributed in the $yz$-plane on the surface of the source sphere. The radiated fields were evaluated throughout a $12 \times 12$ m region in the $yz$-plane centered about the observation sphere center. The normalized error in each observer response was calculated by dividing the $L_2$ norm of the difference between the exact fields and those computed using the PWTD algorithm by the $L_2$ norm of the exact fields. The parameters defining the temporal and spherical interpolation functions, and an estimate of the truncation error introduced by the use of these interpolants, are tabulated in
Table 3.1 along with the average error throughout the observation sphere. It is seen that the error in the observed fields is of the same order as the error introduced by the interpolations. This implies that, as expected, the difference between the analytical and the PWTD solutions depends solely on the error due to interpolation and can be reduced to arbitrary precision. The errors for Cases B, D, F, and H of Table 3.1 are plotted throughout the square observation domain in Figure 3.15. The location of the observation sphere is also depicted in these figures, and it is seen that for all four cases the desired accuracy is obtained throughout the observation sphere.

Note that the structure of the error depicted in Figure 3.15 is quite different than that shown in Figure 3.7. This is due to the fact that the algorithm there relied on a Whittaker-type expansion rather than a finite-cone representation of the field as done here.

Table 3.1: Estimates of relative errors $\varepsilon_t$ and $\varepsilon_s$ introduced by temporal and spatial prolate interpolations, respectively, and the average normalized error in observed fields in the observation sphere.

<table>
<thead>
<tr>
<th>CASE</th>
<th>$\chi_o$</th>
<th>$P_t$</th>
<th>$\chi_1$</th>
<th>$\chi_2$</th>
<th>$P_s$</th>
<th>$\varepsilon_t$</th>
<th>$\varepsilon_s$</th>
<th>Average error</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2.17</td>
<td>4</td>
<td>1.25</td>
<td>2.5</td>
<td>4</td>
<td>$6.16 \times 10^{-3}$</td>
<td>$8.03 \times 10^{-3}$</td>
<td>$1.83 \times 10^{-3}$</td>
</tr>
<tr>
<td>B</td>
<td>2.17</td>
<td>6</td>
<td>1.25</td>
<td>3.2</td>
<td>5</td>
<td>$1.55 \times 10^{-4}$</td>
<td>$1.79 \times 10^{-4}$</td>
<td>$1.99 \times 10^{-4}$</td>
</tr>
<tr>
<td>C</td>
<td>2.17</td>
<td>8</td>
<td>1.25</td>
<td>2.9</td>
<td>7</td>
<td>$5.46 \times 10^{-5}$</td>
<td>$2.02 \times 10^{-5}$</td>
<td>$4.20 \times 10^{-5}$</td>
</tr>
<tr>
<td>D</td>
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<td>11</td>
<td>1.25</td>
<td>3.2</td>
<td>8</td>
<td>$1.48 \times 10^{-6}$</td>
<td>$1.43 \times 10^{-6}$</td>
<td>$2.41 \times 10^{-6}$</td>
</tr>
<tr>
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<td>13</td>
<td>1.25</td>
<td>3.9</td>
<td>8</td>
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<td>$3.60 \times 10^{-7}$</td>
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</tr>
<tr>
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<td>1.25</td>
<td>4.8</td>
<td>9</td>
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<td>$1.11 \times 10^{-8}$</td>
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</tr>
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<td>17</td>
<td>1.25</td>
<td>5.1</td>
<td>9</td>
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<td>$5.15 \times 10^{-9}$</td>
</tr>
<tr>
<td>H</td>
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<td>1.25</td>
<td>5.4</td>
<td>10</td>
<td>$2.22 \times 10^{-10}$</td>
<td>$9.31 \times 10^{-11}$</td>
<td>$3.01 \times 10^{-10}$</td>
</tr>
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</table>
3.6.2 A two-level windowed PWTD enhanced MOT algorithm

Now that the applicability of the windowed PWTD algorithm has been verified for a pair of source and observation groups, we will demonstrate that this algorithm results in lower computational complexities than those obtained by the nonwindowed PWTD algorithm when
applied to the analysis of scattering from large surfaces. The PWTD algorithm is intended to be used in conjunction with time domain integral-equation schemes like MOT. In this subsection, the computational complexity of the two-level windowed PWTD algorithms will be derived. This analysis is akin to that used in illustrating the computational complexity of the frequency domain FMM technique [91, 106].

In order to reduce the computational complexity of the classical MOT algorithm, the scatterer is subdivided into $N_g$ subscatterers or groups, each of which contains approximately $M_s = N_s/N_g$ sources. If two subscatterers are separated by less than a preset distance, these are said to reside in each other's near field, and their interactions are computed using the classical MOT scheme since it becomes impossible to choose a $cT_s/R_s$ of $O(1)$ that satisfies constraints (3.38) and (3.39). All other subscatterer pairs are said to reside in each other's far-field. The cost associated with the computation of the near-fields, $C_{NF}$, is proportional to the square of the number of unknowns per group, the number of groups, and the total number of time steps in the analysis:

$$C_{NF} \propto \left( \text{# of groups} \right) \times \left( \text{# of unknowns per group} \right)^2 \times \text{( # of time steps )}$$

$$\propto N_g M_s^2 N_t$$

$$\propto \left( \frac{N_s}{M_s} \right) M_s^2 N_t$$

$$\propto N_s M_s N_t. \quad (3.71)$$

Interactions between remote groups are accounted for by using the three-stage procedure consisting of aggregation, translation, and disaggregation. During the aggregation stage, a separate set of outgoing subrays is constructed for each source group and for each time interval. During the translation stage, outgoing subrays are convolved with the translation functions on a direction-by-direction basis for each far-field group pair, and the resulting plane waves are added
onto the incoming rays impinging on an observation group. During the disaggregation stage, the fields at the observers are reconstructed by projection of the incoming rays onto the observer points. Note that, as the disaggregation process is the reverse of the aggregation process, its cost is comparable to that of the aggregation step. Let $C_{FF}^{1,3}$ denote the total cost associated with the first and third stages. Then,

$$C_{FF}^{1,3} \propto (\text{# of groups}) \times (\text{# of unknowns per group}) \times (\text{# of ray directions}) \times (\text{# of time steps})$$

$$\propto N_S M_S D_S N_t$$

$$\approx \left(\frac{N_S}{M_S}\right) M_S M_S N_t$$

$$\approx N_S M_S N_t,$$

where it is assumed that $D_S \propto M_S$ as discussed above. The cost of the translation process $C_{FF}^{2}$ is

$$C_{FF}^{2} \propto (\text{# of groups})^2 \times (\text{# of ray directions}) \times (\text{# of time intervals}) \times (\text{cost per subray translation})$$

$$\propto N_S^2 D_S L M'_t \log M'_t$$

$$\propto \left(\frac{N_S}{M_S}\right)^2 M_S N_t \log M_S$$

$$\propto \frac{N_S^2}{M_S} N_t \log M_S,$$

where it was assumed that the translation convolution was performed using an FFT, that $M'_t \propto \sqrt{M_S}$ as discussed earlier, and that $L \propto N_t / M'_t$. The total cost associated with the computation of the fields at all the observers is equal to

$$C_T = C_{NF} + C_{FF}$$

$$= C_{NF} + C_{FF}^{1,3} + C_{FF}^{2}.$$

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It can be verified that for the algorithm described above, the optimal number of unknowns per group is \( M_s \propto N_s^{1/2} \), and the total computation time scales as

\[
C_T \propto N_t N_s^{1.5} \log N_s.
\]  (3.75)

Note that this is the same complexity that can be obtained by the nonwindowed algorithm. However, the cost of the present algorithm can be further reduced if one makes use of the fact that not all plane waves need to be translated from one group to another. Indeed, as the interacting groups move further apart, the number of plane wave components that participate in translation shrinks to a constant.¹ This yields

\[
C_{FF}^2 \propto N_t \frac{N_s^2}{M_s^2} \log M_s.
\]  (3.76)

Using this expression in Eq. (3.74), the optimal value of \( M_s \) is found to be \( N_s^{1/3} \), and the total cost of the field computation scales as

\[
C_T \propto N_t N_s^{4/3} \log N_s.
\]  (3.77)

Note that the computational costs of both the nonwindowed and the windowed two-level PWTD enhanced MOT schemes scale more favorably than those of the classical MOT and FDTD algorithms.

¹ Note that, for source and observation spheres that are not separated by a distance that is very large compared to the sphere radius, the number of rays that need to be translated becomes larger than the constant assumed in the derivation of (3.76). However, it can be shown through a more rigorous derivation that this effect does not alter the above derived complexity estimate for \( C_{FF} \).
3.6.3 A multilevel windowed PWTD enhanced MOT algorithm

To compute the fields using a multilevel strategy, the surface is embedded in a box, which is subdivided into eight child boxes or groups. Each of the nonempty child boxes is again subdivided into eight boxes, and this process is continued recursively until the finest level is reached. At the finest level, each box contains a fixed number of sources that is independent of the problem size. The number of levels $N_l$ is proportional to $\log N_s$. For levels $i = 1, \ldots, N_l$, let $N_g(i)$ denote the number of nonempty groups, $M_s(i)$ the average number of sources in each group, $D_s(i)$ the number of ray directions associated with a group, and $R_s(i)$ the group dimension. Also, the duration of the subray at level $i$, $M_i'(i)$, is assumed to be proportional to $\sqrt{M_s(i)}$. Finally, let $L(i) \propto N_l/M_i'(i)$ denote the number of time intervals associated with level $i$. Assuming that levels are numbered starting from the finest level upward, for a surface scatterer, $N_g(i+1) \propto N_g(i)/4$ and $M_s(i+1) \propto 4M_s(i)$. We also assume that $N_g(1) \propto N_s$, and that $M_s(1)$ and $R_s(1)$ are of $O(1)$. As $i$ increases, $L(i)$ shrinks; hence fewer time intervals are associated with higher levels. Subrays at each level can be constructed by concatenating (partially overlapping) subrays that have been constructed at lower levels. As before, it can be shown that $M_s(i) \propto D_s(i) \propto (R_s(i)/(c\Delta_t))^2$. At any level, two boxes are said to reside in each other’s near field if they are separated by no more than a preset, fixed number of boxes. All other box pairs are said to reside in each other’s far field.

The total cost of the field computation is again a sum of a near-field cost $C_{NF}$ and a far-field cost $C_{FF}$. At the finest level, fields are computed directly for all sources and for all observers that reside in each other’s near field for all time steps. The cost of this operation is
\[ C_{NF} \propto \left( \text{# of finest level groups} \right) \times \left( \text{# of sources per finest level group} \right)^2 \times \left( \text{# of time steps} \right) \]
\[ \propto N_g(1)M_s^2(1)N_t \]
\[ \propto N_sN_t \, . \]

The cost of computing the far-field interactions is again comprised of aggregation, translation, and disaggregation costs. However, in contrast to the two-level algorithm outlined above, these costs are now distributed over all \( N_t \) levels.

First, consider the cost of computing the outgoing rays. At any given level, the outgoing rays associated with a group are constructed by \((i)\) interpolating the spectra of its children to the ray density prescribed by the dimension of their parent group, \((ii)\) concatenating consecutive child subrays two-by-two, and \((iii)\) shifting all ray origins from child box centers to parent box centers. Recall that the last two steps were termed splicing in Section 3.4.2. Similar operations are required when constructing incoming rays. The total cost of constructing the outgoing and incoming rays is

\[ C_{FF}^{1,3} \propto \sum_{i=1}^{N_t} \left\{ \left( \text{# of groups at level } i \right) \times \left( \text{# of directions at level } i \right) \times \left( \text{subray length at level } i \right) \times \left( \text{# of time intervals at level } i \right) \right\} \]
\[ \propto \sum_{i=1}^{N_t} N_g(i)D_s(i)M_s(i)L(i) \]
\[ \propto \sum_{i=1}^{N_t} \left( \frac{N_s}{M_s(i)} \right) M_s(i)N_t \]
\[ \propto N_t N_s \log N_s . \]

Note that in the above estimate the cost of interpolation/anterpolation is assumed to be proportional to the number of directions at level \( i \), \( D_s(i) \propto M_s \), as opposed to the \( O(M_s \log M_s) \) cost derived in Section 3.4.2. This lower complexity can be achieved using local
interpolation/antiprolation schemes for radiated fields over the sphere [51, 99] by a modest (constant) increase in the number of directions due to oversampling.

To compute fields at observers, outgoing rays are translated between all group pairs that reside in each other’s far field and whose parents reside in each other’s near field. The cost of translating subrays is proportional to

\[
C_{FF}^2 \propto \sum_{i=1}^{N_i} \left\{ \left( \text{# of groups at level } i \right) \times \left( \text{# of ray directions at level } i \right) \times \left( \text{# of time intervals at level } i \right) \times \left( \text{cost per subray translation at level } i \right) \right\}
\]

\[
\propto \sum_{i=1}^{N_i} N_g(i) D_s(i) L(i) M^t_1(i) \log M^t_1(i)
\]

\[
\propto N_t N_s \sum_{i=1}^{N_i} \log M_s(i)
\]

\[
\propto N_t N_s M_s(1) \left( \log 2^0 + \log 2^1 + \log 2^2 + \ldots + \log 2^{\log N_s} \right)
\]

\[
\propto N_t N_s (1 + 2 + \ldots + \log N_s)
\]

\[
\propto N_t N_s \log^2 N_s. \tag{3.80}
\]

Comparing the expressions for \(C_{NF}, C_{FF}^{1.3}, \) and \(C_{FF}^2,\) it is seen that the total cost of the multilevel PWTD algorithm is dominated by \(C_{FF}^2;\) hence,

\[
C_T \propto N_t N_s \log^2 N_s. \tag{3.81}
\]

Again, this cost estimate does not make use of the windowed nature of the translation function and can be further reduced if use is made of the truncated version of \(\Psi_n(\theta)\) to form the translation functions (see Subsection 3.5.2). For sufficiently large scatterers, the length of these windowed translation functions scales as \((R_c/c)p_s \Delta \theta \theta_{int}.\) Since, in a multilevel setting, \(R_c\) scales as \(R_s,\) \(\Delta \theta\) scales as \(1/R_s,\) and \(p_s\) and \(\theta_{int}\) remain constant for all levels, the translation function length is of \(O(1).\) Therefore, directly convolving each subray with the translation
function will be more efficient than using an FFT, reducing the cost per subray translation in \( C_{FF}^2 \) from \( O(M'_i(i) \log M'_i(i)) \) to \( O(M'_i(i)) \). This results in a total complexity of

\[
C_T \propto N_t N_s \log N_s.
\]  

(3.82)

3.7 Summary

This chapter introduced two PWTD algorithms and outlined their incorporation into existing MOT algorithms to considerably reduce the computational complexity associated with the evaluation of RTBIs and the analysis of transient surface scattering phenomena. This reduction in cost is achieved through expansion of transient wave fields in terms of a propagating plane wave basis that is characterized by a diagonal translation operator. The translation function derived for the (nonwindowed) PWTD algorithm turns out to be the Fourier transform of the regular part of frequency domain fast multipole method translation function and gives rise to an anticausal ghost signal. Properly sectioning the source signal into subsignals and translating each subsignal separately isolates the ghost signals, which are then time-gated out. The applications of the proposed plane wave expansions on a subsignal basis are formulated as three-stage PWTD algorithms that facilitate ghost-free evaluation of transient fields. It is illustrated both theoretically and experimentally that these algorithms permit the reconstruction of transient fields to arbitrary precision. Incorporation of the (nonwindowed) PWTD algorithm into the classical MOT scheme in a two-level setting permits the analysis of surface scattering phenomena in \( O(N_t N_s^{1.5} \log N_s) \) operations as opposed to \( O(N_t N_s^2) \) operations required by the classical method. The ideas underlying the two-level algorithm are extended into a multilevel framework thereby enabling a further reduction of the computational complexity to \( O(N_t N_s \log^2 N_s) \). Employing the windowed PWTD algorithm further reduces the complexities of the two-level and multilevel schemes to \( O(N_t N_s^{4/3} \log N_s) \) and \( O(N_t N_s \log N_s) \), respectively. The efficacy of these schemes will be demonstrated in the following chapters.
CHAPTER 4
FAST TRANSIENT ANALYSIS OF ACOUSTIC WAVE SCATTERING FROM
RIGID BODIES USING PLANE-WAVE TIME-DOMAIN ALGORITHM

4.1 Introduction

In Section 2.3, a combined field integral equation (CFIE), which relates the velocity potential induced on the surface of a rigid body to the incident field, was derived. A marching-on-in-time (MOT) algorithm to numerically solve this integral equation was also introduced in the same section. In this classical MOT scheme, first, the instantaneous spatial distribution of the wave field over the surface of a scatterer is represented in terms of \( N_s \) (spatial) basis functions. The field amplitudes associated with these basis functions at a time step are calculated from the knowledge of the surface field values of the prior time steps via a retarded time boundary integral (RTBI) and the incident field. Hence, at each time step of an MOT scheme, \( N_s \) field values are evaluated from \( N_s \) past field values incurring a total computational cost that scales as \( O(N_t N_s^2) \) for an analysis that lasts \( N_t \) time steps. It is this high computational complexity that renders the analysis of transient scattering from real size structures unattainable using the MOT method with the current computational resources.

In the previous chapter, PWTD algorithms that can be utilized to reduce the high computational complexity of classical MOT schemes have been introduced. In that chapter, it has been theoretically shown that incorporation of the (nonwindowed) PWTD algorithms into MOT schemes in two-level and multilevel settings reduces the computational complexity of a transient scattering analysis to \( O(N_t N_s^{1.5} \log N_s) \) and \( O(N_t N_s \log^2 N_s) \), respectively. Although, for the sake of simplicity, the scatterer in the previous chapter was assumed to be characterized by a homogeneous Dirichlet boundary condition, i.e., a soft scatterer or a pressure release
surface, it was stated that the results could be easily extended to cases with different boundary conditions.

The purpose of the present chapter is two-fold: First, it demonstrates how the plane wave expansion given in Eq. (3.11), which forms the crux of the three-stage PTWD algorithm, can be adapted to handle different types of sources—or for a scattering analysis different types of boundary conditions. This is accomplished through the introduction of a plane wave expansion for the fields that are related to their sources via the RTBI operator \( \mathcal{L}_c \) introduced in Section 2.3.4. Second, the theoretically predicted computational complexities of the two-level and multilevel PWTD enhanced MOT solvers are verified through numerical experiments in this chapter. The numerical experiments also demonstrate the efficacy of the proposed schemes.

The organization of this chapter is as follows. In the next section, the plane wave expansion associated with the RTBI operator \( \mathcal{L}_c \) is derived and the (nonwindowed) PWTD algorithm for this expansion is outlined. In Section 4.3, incorporation of the PWTD algorithm into the MOT method in a two-level setting is elucidated and the computational complexity of the resulting scheme is derived. Extension of the two-level scheme to a multilevel implementation is presented in Section 4.4. Numerical examples, which verify the applicability and the complexity of the proposed schemes, are presented separately for these two sections. Conclusions for the chapter are stated in Section 4.5.

4.2 The Plane-Wave Time-Domain Algorithm for Acoustic Scattering

As discussed in Section 2.3, acoustic wave scattering from rigid bodies can be analyzed using the integral equation (2.9), which is repeated below for convenience.

\[
(1 - \alpha) \partial_t \varphi^{inc}(\mathbf{r}, t) + \alpha c \hat{n} \cdot \nabla \varphi^{inc}(\mathbf{r}, t) = \mathcal{L}_c \{ \varphi(\mathbf{r}, t) \} \quad \mathbf{r} \in S, \tag{4.1}
\]
where

$$\mathcal{L}_c\{\varphi(r,t)\} = -(1 - \alpha)\mathcal{L}_p\{\varphi(r,t)\} + \alpha c \mathcal{L}_v\{\varphi(r,t)\}$$  \hspace{1cm} (4.2)

$$\mathcal{L}_p\{\varphi(r,t)\} = -\frac{1}{2} \partial_t \varphi(r,t) + \frac{1}{\mathcal{S}} \int_S dr' \varphi(r',t) \cdot \hat{n}' \cdot \nabla' \frac{\delta(t - R/c)}{4\pi R}$$  \hspace{1cm} (4.3)

$$\mathcal{L}_v\{\varphi(r,t)\} = -\frac{1}{\mathcal{S}} \int_S dr' \varphi(r',t) \cdot \hat{n} \cdot \nabla' \frac{\delta(t - R/c)}{4\pi R}.$$  \hspace{1cm} (4.4)

In the above equations, $\varphi^{inc}(r,t)$ and $\varphi(r,t)$ represent the velocity potentials associated with the incident and total fields, respectively. Once $\varphi(r,t)$ on $S$ is computed by solving Eq. (4.1), the scattered field velocity potential $\varphi^{sca}(r,t)$ everywhere can be calculated using

$$\varphi^{sca}(r,t) = \int_S dr' \varphi(r',t) \cdot \hat{n}' \cdot \nabla' \frac{\delta(t - R/c)}{4\pi R}.$$  \hspace{1cm} (4.5)

Discretization of Eq. (4.1) as described in Section 2.3.6 yields the matrix equation

$$\mathbf{Z}_0 \Phi_j = \mathbf{F}_j^{inc} - \sum_{k=1}^{j-1} \mathbf{Z}_k \Phi_{j-k},$$  \hspace{1cm} (4.6)

where—for spatial and temporal basis functions $f_n(r)$, $n = 1, \ldots, N_s$, and $T_j(t)$, $j = 1, \ldots, N_t$, with expansion coefficients $\varphi_{j,n}$, and testing function $\mathbf{f}_m(r)$, $m = 1, \ldots, N_s$—the elements of the vectors $\Phi_j$ and $\mathbf{F}_j^{inc}$ are

$$\Phi_{j,n} = \varphi_{j,n}$$  \hspace{1cm} (4.7)

$$\mathbf{F}_{j,m}^{inc} = \left[ \int_S dr \mathbf{f}_m(r) \left[ (1 - \alpha) \partial_t \varphi^{inc}(r,t) + \alpha c \hat{n} \cdot \nabla \varphi^{inc}(r,t) \right] \right]_{t=T_j}$$  \hspace{1cm} (4.8)
and \( \mathbf{Z}_k \) is an interaction matrix such that the product \( \mathbf{Z}_k \Phi_{j-k} \) yields the velocity potential on \( S' \) at time \( t = t_j \) due to the potential distribution \( \varphi(r, t_{j-k}) \) on \( S \). An element of the \( \mathbf{Z}_k \) matrix is given by

\[
\mathbf{Z}_{k, mn} = \int_{S} dr \bar{f}_m(r) \mathcal{L}_c \{ f_n(r) T_{j-k}(t) \} \bigg|_{t=t_j}.
\]

(4.9)

Numerical solution to Eq. (4.1) is obtained by forming and solving Eq. (4.6) at each time step. Since the interaction matrix \( \mathbf{Z}_0 \) is sparse, solution of the matrix equation is obtained in \( O(\kappa \mathcal{N}) \) operations with a nonstationary iterative solver, where \( \kappa \) denotes the number of iterations. However, evaluation of the summation on the right-hand side of Eq. (4.6) is a computationally expensive task. This operation is equivalent to evaluating \( \mathcal{L}_c \{ \varphi(r, t) \} \), which can be written as a sum over \( N_s \) basis functions, at \( N_s \) observation points and requires \( O(N_s^2) \) operations per time step. Hence, the complexity of a classical MOT analysis for \( N_t \) time steps scales as \( O(N_t N_s^2) \). This cost renders the classical MOT method prohibitively expensive for large-scale scattering analysis.

### 4.2.1 The plane wave expansion

It has been pointed out in the previous section that the most expensive part of the MOT method is evaluating the discrete analogue of

\[
u(r, t) = \mathcal{L}_c \{ \varphi(r, t) \},
\]

(4.10)

which from here on will be referred to as the field radiated by a surface bound source distribution of strength \( \varphi(r, t) \). The computation of this radiated field at multiple observation points can be accelerated if it is represented in terms of a plane wave basis. Next, a plane wave expansion for \( u(r, t) \) is presented that can be evaluated using the three-stage PWTD algorithm of Section 3.3.1.
Consider two fictitious spheres whose centers are located at \( r_s \) and \( r_o \) (Figure 4.1(a)). The radius of each sphere is denoted by \( R_s \) and it is assumed that \(|r_o - r_s| > 2R_s\), i.e., the spheres do not overlap. Assume that a source distribution \( \varphi(r,t) \) resides on a surface \( S' \) enclosed by the first sphere and the field \( u(r,t) \) due to this source distribution is observed inside the second sphere. As the source and observation points can never overlap, Eqs. (4.2)-(4.4) imply that \( u(r,t) \) can be expressed as

\[
u(r,t) = -[(1-\alpha)\partial_t + \alpha c \hat{n} \cdot \nabla] \int_{S'} dr' \varphi(r',t) \ast \hat{n}' \cdot \nabla' \frac{\delta(t-R/c)}{4\pi R}.
\]  

(4.11)

To make the discussion more tractable, the following notation is introduced. The points \( r \) and \( r' \) denote the observer and source locations in the observation and source spheres, respectively. Their positions relative to respective sphere centers are denoted by \( \bar{r}_o = r - r_o \) and \( \bar{r}_s = r' - r_s \). Let \( R_c = r_o - r_s \) denote the vector connecting the source and observation sphere centers, and, without loss of generality, assume it to be aligned with the \( \hat{z} \) axis, i.e., \( R_c = R_c \hat{z} \), where \( R_c = |R_c| \).

As will be evident in the forthcoming analysis, the plane wave representation to be derived is valid for sources of a finite duration \( T_s \) (the maximum of which depends on \( R_c \) and \( R_s \)). However, for signals longer than the maximum allowable \( T_s \), this restriction can be circumvented using the linear nature of the representation. Hence, the first step towards representing the field \( u(r,t) \) as a superposition of plane waves is to break up the source signal \( \varphi(r',t) \) of duration \( T = N_t \Delta t \) into \( L \) overlapping sub-signals \( \varphi_l(r',t) \), \( l = 0, \ldots, L-1 \), each of duration \( T_s \) such that

\[
\varphi(r',t) = \sum_{l=0}^{L-1} \varphi_l(r',t).
\]  

(4.12)
Let $t_i^{\text{min}}$ and $t_i^{\text{max}}$ denote the starting and ending times of the $i^{\text{th}}$ subsignal, i.e., $\varphi_i(r', t) = 0$ outside the interval $t_i^{\text{min}} \leq t < t_i^{\text{max}}$ (Figure 4.2), and let $u_i(r, t)$ denote the field at $r$ due to $\varphi_i(r', t)$; hence, $u(r, t) = \sum_{l=0}^{L-1} u_l(r, t)$.
To arrive at a plane wave representation of $u_l(r,t)$, consider the field $\tilde{u}_l(r,t)$ given by

$$
\tilde{u}_l(r,t) = [(1 - \alpha)\partial_t + \alpha c \hat{n} \cdot \nabla] \frac{\partial t}{8\pi^2 c} \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta \sin \theta \delta(t - \hat{k} \cdot \tilde{r}_0/c) \delta(t - \hat{k} \cdot \tilde{R}_c/c) \tilde{\varphi}_l(\hat{k}, t),
$$

(4.13)

where $\hat{k} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$ and $\tilde{\varphi}_l(\hat{k}, t)$ is the slant stack transform (SST) of the source distribution $\varphi_l(r', t)$ defined as

$$
\tilde{\varphi}_l(\hat{k}, t) = \int_{S'} dr' \varphi_l(r', t) \ast \hat{n} \cdot \nabla' \delta(t + \hat{k} \cdot \tilde{r}_s/c).
$$

(4.14)

The SST maps the source distribution to a time dependent plane wave emanating from the source sphere and propagating along direction $\hat{k}$. Henceforth, plane waves obtained by an SST will be termed outgoing rays. Note that since the outgoing rays are obtained by delaying the subsignal $\varphi_l(r', t)$ by an amount $-\hat{k} \cdot \tilde{r}_s/c$, which can assume values between $-R_s/c$ and $R_s/c$ within a
source sphere, each outgoing ray can span a maximum duration of \( T_s + 2R_s/c \) for a fixed observer.

Substituting Eq. (4.14) into Eq. (4.13), rearranging the order of integrations, and using the fact that \( \mathbf{R} = \mathbf{r}_o + \mathbf{R}_c - \mathbf{r}_s \) yields

\[
\tilde{u}_t(r, t) = -[(1 - \alpha) \partial_t + \alpha c \hat{n} \cdot \nabla] \int_{S'} dr' \varphi_l(r', t) \hat{n}' \cdot \nabla' \tilde{g}(r', r, t), \tag{4.15}
\]

where

\[
\tilde{g}(r', r, t) = -\frac{\delta_t}{8\pi^2 c} \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \delta(t - \hat{k} \cdot \mathbf{R}/c). \tag{4.16}
\]

Note the similarity between Eqs. (4.11) and (4.15). The first step in evaluating the spherical integral in Eq. (4.16) is to transform the integration variables \((\theta, \phi)\) to a new set of angular coordinates \((\theta'', \phi'')\) which are defined with respect to the \(\hat{z}''\) axis aligned with the vector \(\mathbf{R}\) as shown in Figure 4.1(b). In this new coordinate system, \(\hat{k} \cdot \mathbf{R} \rightarrow R \cos \theta''\) and setting \(\tau = (R/c) \cos \theta''\) yields

\[
\tilde{g}(r', r, t) = -\frac{\delta_t}{8\pi^2 R} \int_0^{2\pi} d\phi'' \int_{-R/c}^{R/c} d\tau \delta(t - \tau) \tag{4.17}
\]

\[
= \frac{1}{8\pi^2 R} \int_0^{2\pi} d\phi'' \left[ \delta \left( t - \frac{R}{c} \right) - \delta \left( t + \frac{R}{c} \right) \right]
\]

\[
= \frac{\delta \left( t - \frac{R}{c} \right)}{4\pi R} - \frac{\delta \left( t + \frac{R}{c} \right)}{4\pi R}.
\]

Substituting Eq. (4.17) in Eq. (4.15) yields

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\[
\tilde{u}_l(\mathbf{r}, t) = -[(1 - \alpha) \partial_t + \alpha c \hat{\mathbf{n}} \cdot \nabla] \int_{S'} d\mathbf{r}' \varphi_l(\mathbf{r}', t) \ast \hat{\mathbf{n}}' \cdot \nabla' \frac{\delta(t - R/c)}{4\pi R} \\
+ [(1 - \alpha) \partial_t + \alpha c \hat{\mathbf{n}} \cdot \nabla] \int_{S'} d\mathbf{r}' \varphi_l(\mathbf{r}', t) \ast \hat{\mathbf{n}}' \cdot \nabla' \frac{\delta(t + R/c)}{4\pi R}.
\] (4.18)

In Eq. (4.18), the first term on the right-hand side corresponds to the true observed field \( u_l(\mathbf{r}, t) \) given in Eq. (4.11). Note that, as in Eq. (3.10), the second (anticausal) term represents the ghost signal. Next, how the ghost signal can be removed from \( \tilde{u}_l(\mathbf{r}, t) \) is reviewed, so that the true observed field can be constructed as a superposition of plane waves using Eq. (4.13).

To derive a scheme that retains only the true observed field by time gating \( \tilde{u}_l(\mathbf{r}, t) \), the following observations are in order. From Eq. (4.18), it follows that the ghost signal present in \( \tilde{u}_l(\mathbf{r}, t) \) vanishes after \( t_{l}^{\text{ghost}} = -\min\{R\}/c + t_{l}^{\text{max}} \) and the observed field coincides with the true field for \( t > t_{l}^{\text{ghost}} \). Also, the true field does not reach the observer before \( t_{l}^{\text{trans}} = \min\{R\}/c + t_{l}^{\text{min}} \). Therefore, provided that \( t_{l}^{\text{trans}} > t_{l}^{\text{ghost}} \), the ghost field ceases to exist before the true signal arrives. In addition, if \( t_{l}^{\text{trans}} \geq t_{l}^{\text{max}} \), all source activity related to the \( l \)th time interval ends before the true signal reaches the observer. Hence, it is possible to obtain a ghost-free solution via Eq. (4.13) by choosing \( T_s \leq \min\{R\}/c \), which satisfies both the conditions \( t_{l}^{\text{trans}} > t_{l}^{\text{ghost}} \) and \( t_{l}^{\text{trans}} \geq t_{l}^{\text{max}} \). Consequently, since \((R_c - 2R_s) \leq \min\{R\} \) for arbitrary \( \mathbf{r}' \) and \( \mathbf{r} \) in the source and observation spheres, respectively, the choice \( T_s \leq (R_c - 2R_s)/c \) guarantees that a ghost-free solution can be obtained inside the observation sphere via Eqn (4.13).

In summary, provided \( T_s \leq (R_c - 2R_s)/c \), the field \( u_l(\mathbf{r}, t) \) inside the observation sphere can be expressed as

\[
u_l(\mathbf{r}, t) = \begin{cases} 
0 & ; t < t_{l}^{\text{trans}} \\
\tilde{u}_l(\mathbf{r}, t) & ; t \geq t_{l}^{\text{trans}},
\end{cases}
\] (4.19)
where, by Eqs. (4.13) and (4.14), and the identity \( \nabla \delta(t \pm \mathbf{k} \cdot \mathbf{r}/c) = \pm \mathbf{k} \partial_t \delta(t \pm \mathbf{k} \cdot \mathbf{r}/c)/c \), \( \tilde{u}_I(\mathbf{r}, t) \) is given by

\[
\tilde{u}_I(\mathbf{r}, t) = \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta \sin \theta \left[ (1 - \alpha) - \alpha \hat{\mathbf{n}} \cdot \mathbf{k} \right] \delta(t - \mathbf{k} \cdot \mathbf{r}_o/c) \ast \mathcal{T}(\mathbf{k}, \mathbf{R}_c, t) \\
\ast \int_{S'} d\mathbf{r}' \hat{\mathbf{n}}' \cdot \mathbf{k} \delta(t + \mathbf{k} \cdot \mathbf{r}_e/c) \ast \varphi_I(\mathbf{r}', t)
\]

and the translation function \( \mathcal{T}(\mathbf{k}, \mathbf{R}_c, t) \) is defined as

\[
\mathcal{T}(\mathbf{k}, \mathbf{R}_c, t) = \frac{c^3}{8\pi^2 c^2} \delta(t - \mathbf{k} \cdot \mathbf{R}_c/c).
\]

Equations (4.19)-(4.21) constitute the crux of the three-stage PWTD algorithm for evaluating \( u_I(\mathbf{r}, t) \). In the first stage, outgoing rays are formed by evaluating the SST of the source distribution \( \varphi_I(\mathbf{r}', t) \). The second stage maps the outgoing rays from the source sphere to "incoming" plane waves that impinge upon the observation sphere. This is accomplished by convolving the outgoing rays with the translation function \( \mathcal{T}(\mathbf{k}, \mathbf{R}_c, t) \) at \( t = t^\text{trans}_I \). In the final stage, the incoming rays are projected onto the observer location yielding the true field \( u_I(\mathbf{r}, t) \).

Note that although this three-stage process is similar to that outlined in Section 3.3.1, using the source-field relation given in Eq. (4.11) instead of that used in Chapter 3 (Eq. (3.2)) has altered the forms of the mappings to and from the plane wave basis. The change in the translation function is artificial since the additional time derivatives as well as the scaling constants are actually associated with the first and second stages and are incorporated into the translation function only for a convenient implementation as will be seen shortly.
4.2.2 Practical implementation

In practice, the spectral integration over the unit sphere in Eq. (4.20) cannot be performed over a continuum of directions. Hence, as discussed in Section 3.3.2, \( \bar{u}_l(r, t) \) can be evaluated by using only a finite number of directions if (i) the subsignals \( \varphi_i(r', t) \) are temporally bandlimited and (ii) a spatially bandlimited version of the translation function is used.

The first condition can be met by using the temporarily bandlimited and approximately time-limited interpolation functions \( \psi(t) \) introduced in Section 3.5.2 (see Eq. (3.41)) to generate the subsignals \( \varphi_i(r', t) \) as

\[
\varphi_i(r', t) = \sum_{i=LM_t}^{(l+1)M_t-1} \varphi(r', t_i) \psi(t - t_i) .
\]

(4.22)

Note that, as \( \psi(t) \) can be truncated for \( |t| > p_t \Delta_t \), each subsignal is defined in terms of \( M_t \) samples of the signal \( \varphi(r', t) \) but spans \( M_t + 2p_t \) time steps (Figure 4.2). In other words, whereas each subsignal \( \varphi_i(r', t) \) is formed from samples of \( \varphi(r', t) \) in an interval of length \( M_t \Delta_t \), the duration of each \( \varphi_i(r', t) \) is \( T_s = (M_t + 2p_t) \Delta_t \), and adjacent subsignals overlap by \( 2p_t \) samples. Furthermore, each \( \varphi_i(r', t) \) starts at time \( t_i^{\text{min}} = l(T_s - 2p_t \Delta_t) - p_t \Delta_t \) and vanishes after \( t_i^{\text{max}} = (l+1)(T_s - 2p_t \Delta_t) + p_t \Delta_t \). Since the interpolation functions \( \psi(t) \) are bandlimited to \( \omega_s = \chi_o \omega_{\text{max}} \), where \( \chi_o > 1 \), so is each subsignal.

To meet the second condition, the procedure outlined in Section 3.3.2 can be repeated to show that the spatial spectrum of the translation function can be safely truncated to contain only \( K = \left\lfloor \chi_1 2R_s \omega_s / c \right\rfloor \) harmonics, where \( \chi_1 > 1 \) is an excess bandwidth factor [98, 99], to yield

\[
\mathcal{F}(k, R_c, t) = \begin{cases} 
\frac{\sigma^3}{16\pi^2 c R_c} \sum_{k=0}^{K} (2k + 1)P_k(ct/R_c)P_k(\cos \theta) & ; |t| \leq \frac{R_c}{c} \\
0 & ; \text{elsewhere}
\end{cases} ;
\]

(4.23)
Thus, \( \tilde{u}_i(r, t) \) can be evaluated using \( O(K^2) \) directions via the formula

\[
\tilde{u}_i(r, t) = \sum_{p=0}^{K} \sum_{q=-K}^{K} w_{pq} \left\{ (1 - \alpha) - \alpha \hat{n} \cdot \hat{k}_{pq} \right\} \delta \left[ t - \hat{k}_{pq} \cdot \hat{r}_o / c \right] \star \tilde{\gamma} \left( \hat{k}_{pq}, R_c, t \right) \\
\star \int_{S'} \hat{n}' \cdot \hat{k}_{pq} \delta \left[ t + \hat{k}_{pq} \cdot \hat{r}_s / c \right] \star \varphi_i(r', t) \right\},
\]

(4.24)

where the weights \( w_{pq} \) and the directions \( \hat{k}_{pq} \) are as in Eq. (3.21).

Finally, note that the error incurred by numerically evaluating \( \tilde{u}_i(r, t) \) via Eq. (4.24) depends solely on the choice of the parameters \( \chi_0, \chi_1, \) and \( p_r \), and can be estimated beforehand. As demonstrated in the previous chapter, this error can be reduced to desired precision by varying these three parameters.

4.3 The Two-Level PWTD Enhanced MOT Algorithm

The PWTD algorithm introduced in the previous section does not permit the evaluation of the scattered field observed everywhere on \( S \) since it assumes that the sources producing the field are separated from a spherical region enclosing the observers. However, the PWTD algorithm can be used in tandem with the classical MOT scheme to yield a reduced complexity algorithm as described in Section 3.4.1. Here, this scheme is briefly reviewed and illustrated in Figure 4.3. Note that the acceleration scheme below is independent of the particular discretization used in the MOT scheme because interactions between all sufficiently remote basis/testing functions can be described by pointwise interactions upon application of the proper quadrature rule.
Figure 4.3: Block diagram of the two-level PWTD enhanced MOT algorithm.
4.3.1 The algorithm and its computational complexity

The first step in arriving at a fast algorithm is to divide $S$ into $N_g$ subscatterers, each of which can be enclosed in a sphere of radius $R_s$ (Figure 3.8(a)). It is assumed that each subscatterer, on average, is comprised of $M_s = N_s / N_g$ spatial basis functions. Let $R_{c,\gamma\gamma'}$ denote the distance between the centers of the spheres enclosing subscatterers $\gamma$ and $\gamma'$, $\gamma, \gamma' = 1, \ldots, N_g$. Then, a fundamental subsignal duration $T_s$ of order $R_s/c$ is chosen. The constraint $T_s \leq (R_{c,\gamma\gamma'} - 2R_s)/c$, derived for a two-sphere setting in Section 4.2.1, dictates that the PWTD algorithm can be applied to all sphere pairs for which $R_{c,\gamma\gamma'} \geq 2R_s + cT_s$. Hence, two subscatterers $\gamma$ and $\gamma'$ are said to be in each other’s far field if $R_{c,\gamma\gamma'}$ is larger than $2R_s + cT_s$. If this condition is not met they are said to reside in each other’s near field.

The subsignal duration $T_{s,\gamma\gamma'} \leq (R_{c,\gamma\gamma'} - 2R_s)/c$ for each far-field pair $(\gamma, \gamma')$ is determined such that the outgoing rays associated with this pair can be easily formed by concatenating the rays associated with subsignals of fundamental duration $T_s$. Hence, in accordance with the choice $T_s = (M_t + 2p_t)\Delta_t$, the subsignal duration is defined as $T_{s,\gamma\gamma'} = (M_{t,\gamma\gamma'} + 2p_t)\Delta_t$, where

$$M_{t,\gamma\gamma'} = M_t \left[ \frac{R_{c,\gamma\gamma'} - 2R_s}{cT_s - 2p_t\Delta_t} \right]$$  \hspace{1cm} (4.25)

is an integer multiple of $M_t$.

With this partitioning of the scatterer into subscatterers, the field distribution on $S$ at the $j^{th}$ time step is calculated via the following scheme:

1. Calculate the field on each subscatterer due to sources residing on near-field subscatterers using the interaction coefficients given in Eq. (4.9). Since only a few subscatterers are in the near-field of a given subscatterer, this step requires $O(M_s^2)$ operations per subscatterer.
2. If \( j \) is a multiple of \( M_t \), form outgoing rays of duration \( T_s + 2 R_s / c \) for each subscatterer by evaluating the SST of the source distribution for \( O(K^2) \) directions corresponding to the sampling points discussed in Section 4.2.2. Note that both the number of directions and the number of spatial basis functions per subscatterer scale as the surface area of the sphere. Hence this step can be accomplished in \( O(M_t M_s^2) \) operations, and the average cost per time step is seen to scale as \( O(M_s^2) \).

3. Translate outgoing rays between subscatterer pairs \((\gamma, \gamma')\) whenever \( j \) is a multiple of \( M_t \). This is accomplished by first forming the outgoing rays of duration \( T_s + 2 R_s / c \), by concatenating the rays of duration \( T_s + 2 R_s / c \) generated at Step 2, and then convolving these rays with the translation function \( \tilde{F}(\hat{k}, R_{c,\gamma'}, t) \) given in Eq. (4.23). As each outgoing ray spans a duration of \( T_s + 2 R_s / c \) and \( \tilde{F}(\hat{k}, R_{c,\gamma'}, t) \) is of duration \( 2 R_{c,\gamma'}/c \), the durations of both these signals are proportional to \( M_t \gamma' \) time steps. Hence, it is possible to evaluate the convolution of these two signals in \( O(M_t \gamma' \log M_t \gamma') \) operations using fast Fourier transforms. However, as the translation function \( \tilde{F}(\hat{k}, R_{c,\gamma'}, t) \) is not bandlimited in time, its Fourier transform needs to be evaluated at \( O(M_t \gamma') \) frequencies analytically. To this end, note that the Fourier transform of \( \tilde{F}(\hat{k}, R_{c,\gamma'}, t) \) is given by

\[
\mathcal{F}\{\tilde{F}(\hat{k}, R_{c,\gamma'}, t)\} = \int_{-\infty}^{\infty} \tilde{F}(\hat{k}, R_{c,\gamma'}, t) e^{-j \omega t} dt \\
= \frac{(j \omega)^{3}}{8 \pi^2 c^2} \sum_{k=0}^{K} (2k + 1)(-j)^{k} j_{k}(\omega R_{c,\gamma'}/c) P_{k}(\cos \theta) \\
= \frac{c}{R_{c,\gamma'}^3} \tilde{F}(\hat{k}, \Omega),
\]

where \( \Omega = \omega R_{c,\gamma'}/c \) is the normalized frequency and \( j_{k}(\cdot) \) denotes the spherical Bessel function of order \( k \). Since \( \tilde{F}(\hat{k}, \Omega) \) is bandlimited both in \( \theta \) and \( \Omega \), the Fourier transform
of $\mathcal{F}(\hat{k}, R_{c,\gamma'}, t)$ for any group pair can be evaluated from the samples of $\mathcal{F}(\hat{k}, \Omega)$ in $O(M_{t,\gamma'})$ operations for a given direction. After convolving the outgoing rays with the translation function, the resulting rays are superimposed onto the incoming rays of the observer group.

4. Evaluate the field on each subscatterer due to far-field subscatterers by projecting the incoming rays onto observer locations. This is the reverse operation of the mapping described in Step 2. Hence, it can be accomplished in $O(M_s^2)$ operations per time step.

Clearly, the last three steps implement the three-stage PWTD algorithm in a multiple sphere setting. Note that in this scheme important savings result from superimposing the incoming rays translated from different source spheres before they are projected onto observers.

To arrive at an estimate of the computational complexity of a surface scattering analysis using the resulting algorithm, the cost of performing the above four steps for $N_t$ time steps is studied. The first, second, and fourth steps are carried out for $N_g = N_s/M_s$ subscatterers at an average cost of $O(M_s^2)$ for each of the $O(N_t)$ time steps. Hence, their computational costs scale as $O(N_tN_sM_s)$. In the third step, $N_t/M_{t,\gamma'}$ outgoing rays are translated between $N_g^2$ subscatterer pairs for $O(K^2) = O(M_s)$ directions. This yields a cost that scales as $O(N_tN_s^2M_s^{-1}\log M_{t,\gamma'})$. Noting that $M_{t,\gamma'}$ is proportional to the longest linear dimension of the scatterer, which scales as $O(N_s^{1/2})$ for a surface scatterer, the cost of the third step is seen to scale as $O(N_tN_s^2M_s^{-1}\log N_s)$. Hence, the total computational cost asymptotically scales as $c_1N_tN_sM_s + c_2N_tN_s^2M_s^{-1}\log N_s$, where $c_1$ and $c_2$ are machine and implementation dependent constants. If $M_s$ is chosen proportional to $N_s^{1/2}$, the optimal computational complexity is found to scale as $O(N_tN_s^{1.5}\log N_s)$.  

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4.3.2 Numerical results

In this section, the accuracy and the efficiency of the proposed two-level scheme are demonstrated through numerical analysis of plane wave scattering from several rigid bodies. Two sets of examples are used to validate the accuracy of the two-level scheme. First, scattering from relatively small rigid bodies is analyzed using both the classical MOT scheme and the two-level algorithm, and the temporal variations of the velocity potential at a point on the surface of the scatterer calculated by the two methods are compared. Second, scattering from larger structures, whose analysis using the classical MOT scheme becomes unfeasible, is analyzed using the two-level scheme. Then the far-field signature associated with the surface velocity potential \( \varphi(r, t) \) along a direction \( \hat{k}^s \) given by

\[
\varphi_{\text{far}}(\hat{k}^s, t) = \lim_{r \to \infty} 2\sqrt{\pi} r \varphi^{\text{SCA}}(r\hat{k}^s, t + r/c) \\
= \frac{\delta(t)}{2c\sqrt{\pi}} \int_{S} d\mathbf{r}' (\hat{n}' \cdot \hat{k}^s) \varphi(\mathbf{r}', t + \hat{k}^s \cdot \mathbf{r}' / c)
\]

is calculated for several directions and the scattering cross-section (SCS) of the scatterer along these directions is extracted using a discrete time Fourier transform. Then, the obtained SCS results are compared against SCS data obtained either analytically or by a frequency domain two-level FMM-based boundary element method (FD-BEM). The FD-BEM code, which has been validated independently, uses the same spatial basis functions as the MOT code and circumvents the nonuniqueness difficulty by using the Burton-Miller approach [70]. The computational complexity of the two-level scheme is validated with a final example.

In all the MOT analyses that will be presented, the CFIE in Eq. (4.1) with \( \alpha = 0.5 \) (unless otherwise specified) is solved. The excitation is assumed to be a modulated Gaussian plane wave defined as
\[ \varphi^{inc}(r, t) = \cos\left(\frac{ct - r \cdot \hat{k}^{inc}}{\omega_o/c}\right) \exp\left[-\frac{(ct - r \cdot \hat{k}^{inc})^2}{2\sigma^2}\right] \]  

(4.28)

where \( \omega_o/c \) is the center frequency of the pulse normalized to the wave speed in the medium, \( \sigma \) is a measure of the pulse's temporal width, and \( \hat{k}^{inc} \) is the propagation direction of the incident wave. This pulse can be assumed to be bandlimited to \( \omega_{max} = \omega_o + 6c/\sigma \) since at that frequency the spectral content of the pulse is down by 150 dB from its peak value at \( \omega_o \). Finally, the matrix equation in Eq. (4.6) is solved using a transpose-free quasi-minimal residual algorithm [82]. The initial guess vector for this solver is set equal to the velocity potential vector of the previous time step. With this initial guess, a relative residual error of \( 10^{-6} \) was obtained in \( \kappa < 20 \) iterations even when several thousand spatial basis functions were used.

As a first example, the velocity potential on two spheres illuminated by a Gaussian plane wave characterized by \( \omega_o/c = 0, \sigma = 2.848 \) m, and \( \hat{k}^{inc} = -\hat{z} \) is computed using both the classical and accelerated MOT methods. The scatterer geometry is shown in the inset of Figure 4.4(a). Each sphere was modeled by 262 triangular patches. The parameters of the PWTD algorithm are chosen such that all the interactions between the two spheres are accounted for by the PWTD algorithm. The velocity potential of the scattered field observed at a point on the top sphere is plotted in Figure 4.4(a) for both solution schemes, and the agreement is observed to be excellent. Figure 4.4(b) shows the percent difference between the two solutions normalized to the peak amplitude of the scattered field velocity potential. This error is due to the temporal truncation of the interpolation functions \( \psi(t) \) used in generating the subsignals. As shown in Chapter 3, this error can be reduced to desired precision by properly choosing \( \chi_o, \chi_1, \) and \( p_t \).

As a second example, scattering of a Gaussian plane wave from an almond modeled with 3120 spatial basis functions is analyzed. The almond fits in a box of dimensions \( 5 \times 2 \times 0.5 \) m and is a variant of the almond described in [83]. The parameters of the incident plane wave are
Figure 4.4: Transient scattering of a Gaussian plane wave from two rigid spheres modeled by 524 triangular facets. (a) Velocity potential of the scattered field observed at the top sphere. (b) Percent difference between the classical and accelerated solutions normalized to maximum scattered velocity potential amplitude.

Figure 4.5: Transient scattering of a Gaussian plane wave by a 3120-unknown almond. (a) Total velocity potential observed near the front end. (b) Backscattered far field.

chosen to $\omega_0/c = 0$, $\sigma = 0.7279$ m, and $\hat{k}^{inc} = \hat{x}$. The analysis is performed both with the classical MOT method and the two-level PWTD augmented algorithm. Figure 4.5(a) and Figure 4.5(b) compare the total surface velocity potential at a point near the front end of the almond and
the normalized backscattered ($\hat{k}^s = -\hat{k}^{inc}$) far field signature, respectively. Both figures show perfect agreement between results obtained using the classical and fast methods. The error at the level of $10^{-5}$ incurred by the two-level method is again due to the choice of interpolation parameters and can be reduced as desired. Note that the reflection from the front end and the diffraction from the sharp tip of the almond are accurately captured in the backscattered far-field response.

Next, the efficacy of the fast method for generating SCS data is tested. To this end, a rigid sphere of 1 m radius, modeled with 10106 triangular facets, is illuminated by a Gaussian plane wave, characterized by $\omega_o/c = 2\pi \times 1.458$ rad/m, $\sigma = 2.1836$ m, and $\hat{k}^{inc} = -\hat{z}$. The spectral content of the incident pulse is shown in Figure 4.6(a). The comparisons of the SCS data calculated via the two-level algorithm with those obtained analytically using the Mie series are shown for three different frequencies in Figure 4.6(b-d). Note that although the energy content of the incident pulse at $\omega/c = 2\pi \times 1.166$ rad/m and $\omega/c = 2\pi \times 1.750$ rad/m is 72 dB below the peak at $\omega/c = 2\pi \times 1.458$ rad/m, the SCS is accurately extracted from the time domain response.

Next, scattering from a corner reflector with equal arms of $1 \times 1$ m is considered. As the scatterer is composed of thin surfaces, Eq. (4.1) with $\alpha = 1$ was used for the analysis. The arms are aligned with the $xz$- and the $yz$-planes as depicted in the inset of Figure 4.7(a). The scatterer is modeled using 1600 triangular facets, and it is excited by a pulse for which $\omega_o/c = 2\pi \times 1.02$ rad/m, $\sigma = 1.6378$ m, and $\hat{k}^{inc} = -\hat{x} \cos 30^\circ - \hat{y} \sin 30^\circ$. The temporal variation of the surface velocity potential at the center of the arm in the $xz$-plane is plotted in Figure 4.7(a). Clearly, the values obtained by the classical and fast methods are indistinguishable. As an independent check, the frequency domain solutions to the same problem were obtained at 50 different frequencies and the time domain surface velocity potential was obtained by an inverse Fourier transform. This result, also shown in Figure 4.7(a), validates the MOT results albeit with a lower
precision. In Figure 4.7(b), it is seen that the SCSs in the $xy$-plane are accurately predicted by the MOT scheme at $\omega/c = 2\pi \times 0.729$ rad/m and $\omega/c = 2\pi \times 1.312$ rad/m.

Figure 4.6: Scattering cross-sections of a unit radius sphere calculated through a single transient simulation. (a) Spectrum of the incident pulse. The SCS data at (b) $\omega/c = 2\pi \times 1.166$ rad/m, (c) $\omega/c = 2\pi \times 1.458$ rad/m, and (d) $\omega/c = 2\pi \times 1.750$ rad/m.
Figure 4.7: Transient analysis of scattering from a corner reflector with $1 \times 1$ m arms. (a) Magnitude of surface velocity potential at the center of the arm on the $xz$-plane. (b) Scattering cross-section in the $xy$-plane at $\omega/c = 2\pi \times 0.729$ rad/m and $\omega/c = 2\pi \times 1.312$ rad/m.

As a more realistic example, the SCS of a submarine modeled by 15294 triangular facets (see inset of Figure 4.8(a)) is extracted from a single time domain simulation and the results are compared with those obtained by FD-BEM at $\omega/c = 2\pi \times 0.2915$ rad/m. The submarine is 35 m long, 3.5 m wide, and 6 m high. The parameters of the incident pulse are chosen as $\omega_o/c = 2\pi \times 0.2915$ rad/m, $\sigma = 4.367$ m, and $\hat{k}^{inc} = -\left(\hat{y} + \sqrt{3}\hat{z}\right)/2$. As Figure 4.8(a) and Figure 4.8(b) illustrate, the SCSs in the $xz$- and $xy$-planes are in excellent agreement with those obtained using the frequency domain method.
Figure 4.8: Scattering cross-section of a submarine, modeled with 15294 triangular facets, at \( \omega/c = 2\pi \times 0.2915 \) rad/m in the (a) \( xz \)-plane and (b) \( xy \)-plane.

Finally, the computational complexity of the two-level algorithm is verified by studying scattering from a rigid sphere of unit radius. The sphere was modeled with progressively larger number of triangular facets \((N_s)\) and the maximum frequency component of the incident field was also scaled so that the average edge length of the triangular facets was \(1/10\) of the wavelength at that frequency. The total number of time steps in the analysis was fixed at \(N_t = 500\). The CPU times required to compute the surface velocity potential via the classical method and the PWTD enhanced MOT algorithm on a 360 Mflop Origin2000 computer are shown in Figure 4.9. In this log-log graph, the slopes of the linear lines, which are the "best fitting" lines to the data points in the least squares sense, are 2.0 for the classical MOT method and 1.51 for the PWTD enhanced MOT algorithm. Evidently, the two-level scheme achieves the predicted computational complexity and it becomes advantageous to use this scheme as opposed to the classical MOT scheme for scatterers modeled with more than approximately 1000 spatial unknowns.
4.4 The Multilevel PWTD Enhanced MOT Algorithm

Computational savings beyond those achieved by the two-level PWTD scheme described in the previous section can be attained by casting the PWTD algorithm into a multilevel framework as discussed in Section 3.4.2. Just like in the two-level scheme, multilevel PWTD schemes assume that the scatterer is partitioned into a large set of subscatterers, and interactions between nearby subscatterers are accounted for classically. Contrary to the two-level algorithm, however, subscatterers are hierarchically aggregated into composite entities and the PWTD scheme is only invoked to account for interactions between composite subscatterers when continued aggregation would render the PWTD scheme inapplicable. The efficiency of the proposed multilevel scheme hinges on that of the operations that establish communication between the different levels in the hierarchical structure. In this section, first the notation related to the multilevel framework will be introduced. Then, the multilevel fast algorithm will be briefly outlined and its computational complexity and memory requirements will be estimated. Finally, numerical examples will be presented that validate these estimates and demonstrate the efficacy of the proposed algorithm.
4.4.1 Notation

A multilevel subdivision of the scatterer $S$ is accomplished by recursively subdividing a fictitious cubical box that encloses $S$. Initially, this box is divided into eight boxes of equal size. Each resulting "child" box is recursively subdivided into eight smaller boxes until the linear dimensions of the smallest boxes so obtained measure approximately one wavelength at $\omega_{\text{max}}$ (or a fixed fraction thereof). These finest boxes are termed "level 1 boxes," and the collection of spatial basis functions that fall into a level 1 box is said to form a "level 1 group" (or a "level 1 subscatterer"). Higher level boxes and groups are similarly defined. For levels $i = 1, \ldots, N_l$, let $N_g(i)$ denote the number of groups (nonempty boxes), $M_s(i) = N_s/N_g(i)$ the average number of spatial basis functions per group, $R_s(i)$ the radius of the sphere that encloses a level $i$ box, and $K(i) = \left[ 2 \pi R_s(i) \omega_{\text{max}} / c \right]$ the number of spherical harmonics to be used in Eqs. (4.23) and (4.24) whenever a level $i$ box participates in a PWTD translation. From Eq. (4.24) it is evident that $D(i) = (K(i) + 1)(2K(i) + 1)$ directions are required to properly represent fields at level $i$. It should be noted that, for a surface scatterer, $M_s(i)$ and $K(i)$ scale proportional to $(R_s(i))^2$ and $R_s(i)$, respectively, and therefore $D(i)$ is of $O(M_s(i))$. Also, as the linear dimensions of the level 1 boxes are chosen proportional to the wavelength at $\omega_{\text{max}}$, it follows that $(R_s(1)\omega_{\text{max}}/c)$ and $M_s(1)$ are of $O(1)$, that $N_g(1) \propto N_s$, and that $N_l \propto \log N_s$. Finally, note that, for a surface scatterer, $N_g(i + 1) \approx N_g(i)/4$ and $M_s(i + 1) \approx 4M_s(i)$.

In the two-level algorithm of the previous section, all sufficiently remote group pairs interact, i.e., exchange information through a PWTD translation, at the lowest level. This is not so in the multilevel algorithm, where lowest level groups are recursively aggregated into larger entities. Indeed, as long as the spheres circumscribing these composite groups are sufficiently remote, it turns out to be more efficient to have composite structures interact rather than to have their constituents exchange information individually. Therefore, it is important to identify all
group pairs that interact at a given level (this is a subset of all group pairs that exist at that level). To this end, the fundamental subsignal duration at level \( i \) is defined as \( T_s(i) = (M_t(i) + 2p_t)\Delta_t \) with \( M_t(i) = 2^{(i-1)} M_t(1) \) and \( M_t(1) \propto R_s(1)/(c\Delta_t) \). The constraint that \( T_s \leq (R_c - 2R_s)/c \), stated in Section 4.2.1 for a two-sphere setting, dictates that the PWTD algorithm can be applied to sphere pairs at level \( i \) whose centers are separated by at least \( 2R_s(i) + cT_s(i) \). Hence, starting at level \( N_I \), all box pairs whose centers are separated by more than \( 2R_s(N_I) + cT_s(N_I) \) are identified as “level \( N_I \) far-field pairs.” Then, all level \( N_I - 1 \) group pairs with group centers separated by more than \( 2R_s(N_I - 1) + cT_s(N_I - 1) \) and describing interactions that have not yet been accounted for by any of the level \( N_I \) pairs are identified as “level \( N_I - 1 \) far-field pairs.” This process is continued and far-field pairs are identified at each level (including level 1) as those pairs that are considered well separated at a given level and that have not yet been accounted for at a higher level. The level 1 pairs with group centers separated by less than \( 2R_s(1) + cT_s(1) \) are classified as near-field pairs. In this way, each and every source/observer (basis/testing function) combination belongs to one and only one pair that is classified as either a far-field or a near-field pair. Also, distant source/observer combinations tend to belong to higher level far-field pairs than those that reside close to one another. This classification of group pairs was illustrated in Figure 3.8(b).

### 4.4.2 The algorithm and its computational requirements

Now that requisite notation has been introduced, an algorithm for efficiently evaluating the summation in Eq. (4.6) can be outlined. Note, once again, that the evaluation of this summation is equivalent to testing by \( N_s \) spatial testing functions the field \( \mathcal{L}_c \{ \phi(r, t) \} \) generated by \( N_s \) spatial basis functions at time \( t = t_j \).
1. Evaluation of the near-field interactions: For each near-field group pair \((\gamma, \gamma')\), the tested fields in group \(\gamma\) are calculated classically, i.e., without invoking the PWTD scheme, as

\[
\sum_{k=1}^{j-1} \overline{Z}_k^{\gamma'} \Phi_{j-k}^{\gamma'},
\]

where \(\overline{Z}_k^{\gamma'}\) is the matrix made up of those elements of the matrix \(\overline{Z}_k\) that relate the fields over group \(\gamma\) to the sources residing in group \(\gamma'\), and \(\Phi_{j-k}^{\gamma'}\) is a vector comprised of the coefficients \(\varphi_{j-k,n}\) for all \(n\) in group \(\gamma'\). The multilevel partitioning of the scatterer constructed earlier guarantees that all the near-field pairs reside at level 1 and that each level 1 group participates in only a small number of near-field interactions. Hence, at each time step, the evaluation of the sum (4.29) requires \(O(N_s)\) operations since there are \(O(N_s)\) near-field pairs with \(O(1)\) interactions per pair.

2. Evaluation of the far-field interactions: All interactions between a pair of far-field groups \((\gamma, \gamma')\) are calculated via the three-stage PWTD algorithm:

a) When the time step is a multiple of \(M_t(i)\), the outgoing rays emanating from groups at level \(i\) are generated. For level 1 groups, this is accomplished by numerical evaluation of the integral over \(S'\) in Eq. (4.24). Since this operation maps the signal over each of \(N_s\) sources to \(D(1) \propto O(1)\) directions, outgoing rays at level 1 can be constructed in \(O(N_s)\) operations per time step. At higher levels, the outgoing rays of a group are calculated from those of its children using the interpolation and splicing operations introduced in Section 3.4.2 at a cost that scales as \(O(N_s \log^2 N_s)\) per time step.

b) Once the outgoing rays at level \(i\) have been generated, translation between each level \(i\) far-field pair \((\gamma, \gamma')\) can be performed. This is accomplished by convolving the outgoing rays of the source group \(\gamma'\) with the appropriate translation function and superimposing

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the resulting rays onto the incoming rays of the observer group $\gamma$. The duration of both the outgoing rays and the translation functions are proportional to the box size at the level with which they are associated and hence are described by $O(M_f(i))$ samples. Therefore, as explained in Section 4.3, it is possible to evaluate the convolution of these two signals in $O(M_f(i) \log M_f(i))$ operations using fast Fourier transforms and the translation function tables

$$
\mathcal{T}_f(\hat{k}, \Omega) = \frac{R_{c,\gamma'}^3}{c} \mathcal{F} \{ \mathcal{T}(\hat{k}, R_{c,\gamma'}, t) \} = \frac{(j\Omega)^3}{8\pi^2} \sum_{k=0}^{K(i)} (2k+1)(-j)^k j_k(\Omega) P_k(\cos \theta),
$$

where $\Omega = \omega R_{c,\gamma'}/c$ is the normalized frequency, $\theta = \hat{k} \cdot R_{c,\gamma'}/R_{c,\gamma'}$, and $j_k(\cdot)$ denotes the spherical Bessel function of order $k$. Since $\mathcal{T}_f(\theta, \Omega)$ are bandlimited both in $\theta$ and $\Omega$, the Fourier transform of $\mathcal{T}(\hat{k}, R_{c,\gamma'}, t)$ for any group pair at level $i$ can be evaluated from the samples of $\mathcal{T}_f(\theta, \Omega)$ in $O(M_f(i))$ operations for a given direction. As the translation operation is repeated for $N_f/M_f(i)$ time intervals, for $D(i) \approx M_s(i)$ directions per far-field pair, for $O(N_g(i))$ pairs per level, and for all $\log N_s$ levels, the overall cost of this step scales no worse than $O(N_f N_s \log^2 N_s)$.

c) When the time step is a multiple of $M_f(i)$ for a level $i$, the incoming rays at that level are obtained by resecting and interpolating the incoming rays at a higher level. This process has to be carried out starting from the highest level, as the incoming rays at a level have to be formed completely before they are resected to form the rays at lower levels. Finally, the tested fields are evaluated by convolving the incoming rays at level 1 with
\[
\int_S dr \tilde{f}_m(r) \left[ (1-\alpha) \partial_t - \alpha \hat{n} \cdot \hat{k}_{pq} \right] \delta \left( t - \hat{k}_{pq} \cdot (r-r_c)/c \right),
\]  
(4.31)

where \( r_c \) is the center of the box that contains \( \tilde{f}_m(r) \), and summing over all level 1 ray directions \( \hat{k}_{pq} \) as in Eq. (4.24). This step, which is conceptually the transpose of forming the outgoing rays, can also be accomplished in \( O(N_t N_s \log^2 N_s) \) operations.

Summing up the cost of each step, it is seen that the computational cost of this algorithm, which evaluates the summation in Eq. (4.6) \( N_t \) times, scales as \( O(N_t N_s \log^2 N_s) \). This computational complexity is significantly lower than the \( O(N_t N_s^2) \) complexity of the classical MOT algorithms.

It is important to point out that the reduction in the computational complexity in the multilevel algorithm is not achieved at the expense of increased memory requirements. As the surface field values throughout the analysis have to be stored in any MOT scheme, the essential storage requirements associated with all MOT schemes scale as \( O(N_t N_s) \). In a classical MOT implementation, one can either only require \( O(N_t N_s) \) storage by calculating the \( O(N_s^2) \) nonzero elements of the \( \overline{Z}_k \) matrices every time step, or compute these matrix elements once and for all and store them. The first approach, although highly memory efficient consumes an exorbitant amount of computation time, even for very small problems. On the other hand, the second approach reduces computation time at the cost of \( O(N_s^2) \) memory requirements. As the focus of this paper is to minimize the computation time, the second approach is adopted in the section on numerical results for comparison purposes. As for the memory requirements of the multilevel algorithm, it can be shown that the storage of the incoming and outgoing rays dominates the memory complexity. The memory required for storing the rays at level \( i \) is proportional to the number of groups at that level, \( N_g(i) = N_s/M_s(i) \), the number of ray directions per group, \( D(i) \propto M_s(i) \), and the total duration of the rays stored per direction,
$O(M_t(i))$. Hence the amount of storage per level is $O(M_t(i)N_s)$. Noting that $M_t(N_t) \propto N_t$ and making use of the fact that $M_t(i-1) = 0.5M_t(i)$ yields an overall memory complexity of $O(N_tN_s)$, which scales as the essential amount of required storage. Note that for a typical surface scatterer, $N_t$ typically scales as $O(N_s^{0.5})$, and in such cases the derived memory complexity is of $O(N_s^{1.5})$ as opposed to $O(N_s^2)$ for the classical scheme (when all interactions are stored).

### 4.4.3 Numerical results

In this section, the accuracy and efficiency of the proposed multilevel PWTD enhanced MOT (ML-PWTD) scheme are demonstrated by applying it to analysis of plane wave scattering from several rigid bodies. The ML-PWTD scheme is validated by comparing the results obtained using this method either directly to those obtained using an MOT scheme, or indirectly, upon Fourier transformation, to those obtained using a frequency domain boundary element method (FD-BEM). The FD-BEM code uses the same spatial basis functions as the ML-PWTD scheme, circumvents the nonuniqueness difficulty by using the Burton-Miller approach, and has been validated independently. Since all the scatterers analyzed in this section are closed bodies, $\alpha$ in Eq. (4.1) is set to 0.5. In all time domain simulations, the excitation is a modulated Gaussian plane wave defined as

$$\varphi^{inc}(\mathbf{r}, t) = \cos \left( (ct - \mathbf{r} \cdot \hat{\mathbf{k}}^{inc}) 2\pi f_o/c \right) \exp \left[ -\left( ct - \mathbf{r} \cdot \hat{\mathbf{k}}^{inc} - 6\sigma \right)^2 / 2\sigma^2 \right], \quad (4.32)$$

where $f_o$ is the center frequency of the pulse, $\sigma$ is a measure of the pulse's temporal width, and $\hat{\mathbf{k}}^{inc}$ is its propagation direction. This pulse safely can be assumed to be bandlimited to $\omega_{max} = 2\pi f_o + 6c/\sigma$ since at that frequency the spectral content of the pulse is down by more than 150 dB from its peak value at $f_o$. 

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As a first example, plane wave scattering from an almond (depicted in the inset of Figure 4.10(a)) is analyzed. The almond, which is modeled by 8044 triangular facets, is a scaled version of that described by Woo et al. [83] with a total length of 10 m. The parameters of the incident wave are chosen as $f_\alpha/c = 0.4373 \text{ m}^{-1}$, $\sigma = 2.9776 \text{ m/rad}$, and $\hat{k}^{inc} = \hat{\mathbf{i}}$. The surface velocity potential is calculated using both the classical MOT method and the ML-PWTD scheme. In the multilevel scheme, approximately 76% of the interactions were calculated by the PWTD algorithm with three levels, and the parameters that control the accuracy were set to $p_t = 4$ and $\chi = 5.0$. The absolute value of the surface velocity potential near the round end calculated by both methods is plotted on a logarithmic scale with respect to normalized time $ct$ in Figure 4.10(a) and excellent agreement is observed. The $L_2$ norm of the difference of the two results divided by the $L_2$ norm of the one calculated by the classical method is found to be $3.99 \times 10^{-4}$. This difference can be made smaller by increasing $p_t$ and $\chi$. This test demonstrated the accuracy of the near field computed using the ML-PWTD scheme at a given point on the scatterer. To check the accuracy of the calculated fields all over the scatterer.

![Figure 4.10: Scattering from an almond. (a) Surface velocity potential $\varphi(r, t)$ near the round tip of the almond. (b) Back scattered far field signature.](image)

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surface, the far-field signature associated with the surface velocity potential $\varphi(r, t)$ along a direction $\hat{k}^s$ given by Eq. (4.27) is calculated in the backscatter direction $\hat{k}^s = -\hat{z}$ (Figure 4.10(b)). Again, the agreement between the far-field signatures calculated via the classical and PWTD enhanced codes is excellent.

Figure 4.11: (a) SCS pattern of a unit radius sphere for $f/c = 1.822$ m$^{-1}$. (b) Percent error in SCS over a range of frequencies.

Next, the efficacy of the fast method in generating SCS data over a broad range of frequencies is inspected. To this end, a sphere of radius 1 m is modeled with 25246 triangular patches and is excited by a wave with $f_o/c = 1.166$ m$^{-1}$, $\sigma = 0.9358$ m/rad, and $\hat{k}^{inc} = -\hat{z}$. The evolution of the velocity potential at the surface of the sphere is computed by the ML-PWTD scheme, and the far-field signatures are evaluated using Eq. (4.27) for a number of elevation angles. Then, the SCSs of the scatterer along these directions are extracted at particular frequencies from the far-field signatures via a discrete time Fourier transform and compared to analytical values obtained using a Mie series. One such comparison, at $f/c = 1.822$ m$^{-1}$, is shown in Figure 4.11(a). It is seen that the agreement between the MOT result and the analytical one is good, even though the incident signal power at this frequency is 64.6 dB lower.
than the power at the center frequency. The $L_2$ norm of the error in the SCS at different frequencies normalized to the $L_2$ norm of the SCS at each frequency is plotted in Figure 4.11(b). It is seen that the time domain simulation provides accurate far field information over a broad range of frequencies.

In the next example, the PWTD enhanced MOT scheme is pushed to its limits insofar as the computational resources are concerned. The scatterer (depicted in the inset of Figure 4.12(a)) is a $10 \times 3 \times 1$ m rectangular box modeled using 107500 triangular facets, and the parameters of the incident field are chosen as $f_o/c = 1.365$ m$^{-1}$, $\sigma = 1.048$ m/rad, and $\hat{k}^{inc} = -\hat{z}$. Approximately 98% of the interactions were handled by the multilevel PWTD algorithm, which utilized five levels for this case. The results obtained by this simulation are validated by comparing the SCS patterns extracted at different frequencies to those obtained using the FD-BEM code. Figure 4.12 shows how the SCS data in the $xy$- and $xz$-planes generated by the two methods compare at $f/c = 0.8746$ m$^{-1}$ and $f/c = 1.80$ m$^{-1}$. Note that these frequencies are near the edges of the frequency band occupied by the incident pulse as the incident power levels at these frequencies are 45.28 dB and 35.63 dB below the power at the center frequency. At both frequencies, the agreement is very good in spite of the low power levels. Similar tests were carried out at several other frequencies with higher incident field powers, and, as expected, even better agreement between the ML-PWTD and FD-BEM results was observed.

As a more realistic example, the SCS of a submarine was calculated for an incident wave characterized by $f_o/c = 0.4373$ m$^{-1}$, $\sigma = 3.275$ m/rad, and $\hat{k}^{inc} = -0.224\hat{x} - 0.8365\hat{y} - 0.5\hat{z}$. The scatterer, which is 4.432 m wide, 7.585 m tall, and 46 m long, is modeled with 91366 triangular facets and oriented as depicted in the inset of Figure 4.13(a). The model is designed to include a variety of generic surface shapes like a spheroidal cap at the front end and a conical tip at the tail as well as smooth surfaces and wedge-like structures at the top. The SCS patterns in
the plane that includes the \( \hat{k}^{inc} \) and \( \hat{z} \) vectors are plotted in Figure 4.13 for \( f/c = 0.2624 \text{ m}^{-1} \) and \( f/c = 0.585 \text{ m}^{-1} \). In these plots, the angle \( \zeta \) is measured from the +z axis, and the angle \( \zeta = 120^\circ \) indicates the forward scattering directions. Again, agreement between the results obtained by the two methods is excellent.

Figure 4.12: SCS patterns of a 10 \( \times \) 3 \( \times \) 1 m box modeled with 107500 triangular facets in (a) xy-plane at \( f/c = 0.8746 \text{ m}^{-1} \), (b) xz-plane at \( f/c = 0.8746 \text{ m}^{-1} \), (c) xy-plane at \( f/c = 1.80 \text{ m}^{-1} \), (d) xz-plane at \( f/c = 1.80 \text{ m}^{-1} \).
Figure 4.13: SCS patterns of an obliquely insonified submarine modeled with 91366 triangular facets at (a) $f/c = 0.2624 \text{ m}^{-1}$ and (b) $f/c = 0.585 \text{ m}^{-1}$.

Finally, the computational and memory complexities of the proposed ML-PWT scheme are verified. To this end, the $10 \times 3 \times 1 \text{ m}$ rectangular box was first modeled with 1548 triangular facets ($N_s$), and this number was increased progressively to as large as $N_s = 107500$. In addition, the center frequency and width of the incident pulse were scaled such that the average edge length of the triangular facets was 1/10 of the wavelength at the highest frequency. The surface velocity potential was calculated in each case for $N_t = 500$ time steps. The CPU times and the amount of memory required to complete the analysis using the classical and the proposed schemes are shown in Figures 4.14(a) and 4.14(b), respectively. Also, to serve as a reference, the curve $C_c N_s \log^2 N_s$ is plotted in Figure 4.14(a), as is the curve $C_m N_s$ in Figure 4.14(b). The constants $C_c$ and $C_m$ are chosen so that the reference curves match the values for the PWTD enhanced method at $N_s = 38700$. Evidently, the multilevel PWTD enhanced MOT scheme performs with the predicted complexities and becomes more advantageous to employ when $N_s$ exceeds approximately 1600.
Figure 4.14: (a) Computational and (b) memory requirements of the classical MOT and ML-PWTD schemes.

4.5 Conclusions

This chapter elucidated the implementations of two-level and multilevel PWTD enhanced MOT solvers for analyzing transient scattering from rigid bodies. The computational cost of performing a scattering analysis using the proposed two-level and multilevel algorithms was shown to scale as $O(N_t N_s^{1.5} \log N_s)$ and $O(N_t N_s \log^2 N_s)$, respectively, both of which are considerably lower than the $O(N_t N_s^2)$ complexity of the classical MOT methods. It was also shown that the memory requirements of the multilevel PWTD scheme scale as $O(N_t N_s)$. Numerical experiments indicate that the two-level PWTD enhanced MOT scheme becomes more efficient than the classical MOT method when $N_s \geq 1000$, whereas for the ML-PWTD the breakeven point is at $N_s \approx 1600$. As verified by the numerical results, the PWTD-enhanced MOT schemes put the accurate characterization of broadband scattering from large, realistic objects within the reach of current computing resources.
CHAPTER 5
FAST TRANSIENT ANALYSIS OF ELECTROMAGNETIC WAVE SCATTERING FROM CONDUCTING BODIES USING PLANE-WAVE TIME-DOMAIN ALGORITHM

5.1 Introduction

Just like their acoustical counterparts, classical MOT methods for analyzing electromagnetic wave scattering from PEC bodies also suffer from exorbitant computational requirements. For example, the classical MOT algorithm outlined in Section 2.4 for solving the time domain CFIE introduced in the same section has a computational complexity of $O(N_t N_s^2)$ when the current induced on the scatterer is modeled in terms of $N_s$ spatial and $N_t$ temporal basis functions. As was done in the previous chapter for the acoustical case, the classical MOT codes for analyzing electromagnetic phenomena can also be supplemented by the (nonwindowed) PWTD algorithm introduced in Section 3.3 to reduce this high computational complexity. Again, the computational costs of the two-level and multilevel implementations of the PWTD enhanced MOT algorithms are expected to scale as $O(N_t N_s^{1.5} \log N_s)$ and $O(N_t N_s \log^2 N_s)$, respectively. Furthermore, since the electromagnetic problem is a vector wave problem—with each component of the vector field satisfying the scalar wave equation—it is expected that a blatant application of the scalar PWTD algorithm to each field component would require three times the work required by the acoustic implementation. However, a closer look at the three-stage PWTD algorithm suggests further savings that reduce the "constant in front" of the final complexities. The key observation enabling these savings is that the outgoing (incoming) rays associated with a group of sources (observers) represent the radiation (reception) pattern associated with that group. Since the far field pattern of an electromagnetic source or observer distribution is completely characterized by its angular components, each plane wave in
the PWTD algorithm can be specified with only two components. Alternatively, the same conclusion can be arrived at by realizing that the PWTD algorithm relies on a decomposition of wave fields into homogeneous (propagating) plane waves and that the field vectors are perpendicular to the propagation direction for a homogeneous electromagnetic plane wave. Since the propagation direction of each ray in the PWTD algorithm is known, it suffices to carry out the calculations using only the two components of the fields perpendicular to the propagation direction for each ray.

In this chapter, these ideas will be put together to implement efficient PWTD enhanced MOT schemes for analyzing transient scattering from PEC bodies. To this end, first, the plane wave expansion, which is amenable to implementation as a three-stage PWTD algorithm, for the retarded time boundary integral associated with the CFIE of Section 2.4.4 will be derived. Then, how this expansion can be utilized in two-level and multilevel settings to reduce the computational complexity of an MOT analysis of scattering from a PEC target will be elucidated in Sections 5.3 and 5.4, respectively. Numerical examples verifying the predicted computational complexity and the efficacy of each scheme will also be presented in these sections. Conclusions will be stated in Section 5.5.

5.2 Plane-Wave Time-Domain Algorithm for Electromagnetic Scattering

As discussed in Section 2.4, electromagnetic wave scattering from PEC bodies can be analyzed using the integral equation (2.38), which can be stated as

$$\mathcal{V}_c\{\mathbf{E}^i(r, t), \mathbf{H}^i(r, t)\} = \mathcal{L}_c\{\mathbf{J}(r, t)\} \quad r \in S_-,$$

(5.1)

where

$$\mathcal{V}_c\{\mathbf{E}^i(r, t), \mathbf{H}^i(r, t)\} = -(\beta / \eta_0)\hat{n} \times \hat{n} \times \mathbf{E}^i(r, t) + \hat{n} \times \mathbf{H}^i(r, t),$$

(5.2)
\[ L_c \{ J(r,t) \} = -\frac{\beta}{\eta_0} L_e \{ J(r,t) \} + L_h \{ J(r,t) \}, \] 

\[ L_e \{ J(r,t) \} = \hat{n} \times \hat{n} \times \frac{\mu_0}{4\pi} \int_0^t \int_S \frac{d^2}{d^2 t'} \left( 2 \nabla \times \frac{J(r',t' - R/c)}{R} \right) \cdot A(r'), \] 

\[ L_h \{ J(r,t) \} = -\frac{1}{4\pi} \hat{n} \times \int_S d\mathbf{r}' \nabla \times \frac{J(r',t - R/c)}{R}. \]

In the above equations, \( S_+ \) denotes a surface just inside the scatterer's surface \( S \), \( \beta \) is a positive real constant, and \( \mathbb{I} \) is the identity dyad. The form of \( L_c \{ J(r,t) \} \) in Eq. (5.4) can be derived using the relation

\[ E^s(r,t) = -\int_0^t \left( \nabla \times \frac{J(r',t - R/c)}{R} \right) \cdot A(r',t') \] 

instead of Eq. (2.31). Discretizing Eq. (5.1) using the MOT method described in Section 2.4.6—i.e., expanding the currents as in Eq. (2.44) and applying Galerkin testing to both sides of the resulting equation at \( t = t_j = j\Delta_t \)—results in the matrix equation

\[ \mathbf{Z}_0 \mathbf{I}_j = \mathbf{F}_j^{inc} - \sum_{i=1}^{j-1} \mathbf{Z}_i \mathbf{I}_{j-i}, \] 

where \( \mathbf{I}_j \) is an array of the weights \( I_{n,j} \), \( n = 1,\ldots,N_s \),

\[ \mathbf{F}_{j,m}^{inc} = \left\langle f_m(r), \mathcal{V}_c \{ E^l(r,t), H^l(r,t) \} \right\rangle_{t=t_j}, \] 

and

\[ \mathbf{Z}_{i,mn} = \left\langle f_m(r), L_c \{ f_n(r) T_{j-i}(t) \} \right\rangle_{t=t_j}. \]
Equation (5.7) constitutes the basis of the classical MOT scheme; it relates the currents on the surface at $t = t_j$ to those at $t \leq t_{j-1}$, and hence permits the recursive computation of currents for all times. The dominant cost in the construction and solution of Eq. (5.7) at each time step is the calculation of the vector sum that appears on its right-hand side. The cost of evaluating this sum scales as $O(N_s^2)$; indeed, the field measured by each testing function contains contributions from all $N_s$ basis functions. Since this sum is evaluated for all $N_t$ time steps, the total cost of this analysis scales as $O(N_t N_s^2)$. In the next section, a succinct derivation of the basic PWTD algorithm for efficient computation of electromagnetic fields will be presented.

5.2.1 The plane wave expansion

From the preceding discussion, it is apparent that computing interactions between individual basis functions, as in traditional MOT schemes, leads to a computationally inefficient algorithm. This suggests that one of the keys to constructing a reduced-complexity algorithm is development of schemes that permit the computation of interactions in a group-wise manner. Hence, the first task is to develop the PWTD algorithm that achieves efficient computation of group-wise interactions. To this end, consider two fictitious boxes, each of which can be enclosed in a sphere of radius $R_s$ as shown in Figure 5.1. One of these boxes contains the basis function $f_n(r)$ and is dubbed the source box; the other contains the basis function $f_m(r)$ and is called the observation box. The centers of these boxes are located at $r_n^c$ and $r_m^c$, respectively. The vector connecting the centers of these boxes is denoted by $R_c = r_m^c - r_n^c$ and $R_c = |R_c|$. For the purpose of this exposition, let the current associated with the source basis function be characterized as

$$J_n(r,t) = \sum_{j=1}^{N_t} I_{n,j} f_n(r) T_j(t) = f_n(r) f_n(t).$$

\[ (5.10) \]
The time signature \( f_n(t) \) is divided into \( L \) consecutive subsignals \( f_{n,l}(t) \), each of duration \( T_s = (M_t + 1)\Delta_t \) (with \( LM_t = N_t \)) occupying a time slice \( t_l^{\text{min}} \leq t \leq t_l^{\text{max}} \) for \( l = 1, \ldots, L \), where \( t_l^{\text{min}} = (l-1)M_t\Delta_t \), and \( t_l^{\text{max}} = lM_t\Delta_t + \Delta_t \) (see Figure 5.2(a)). In keeping with this division, the current source \( J(r,t) \) can be rewritten as

\[
J_n(r,t) = \sum_{l=1}^{L} J_{n,l}(r,t), \quad (5.11)
\]

where

\[
J_{n,l}(r,t) = f_n(r)f_{n,l}(t) \quad ; \quad f_{n,l}(t) = \sum_{j=(l-1)M_t+1}^{lM_t} I_{n,j} T_j(t). \quad (5.12)
\]

Then, the vector potential associated with one subsignal is

\[
A_{n,l}(r,t) = \frac{\mu_0}{4\pi} \int_{S_n} dr' \frac{f_n(r')f_{n,l}(t - R/c)}{R}. \quad (5.13)
\]
Alternatively, motivated by frequency domain fast multipole methods, which rely heavily on plane wave expansions, consider the field

$$\tilde{A}_{nl}(r,t) = -\frac{\mu_0 c}{8\pi^2 c} \int d^2 k \int dr' f_n(r') \delta \left(t - \frac{\hat{k} \cdot (r - r')}{c}\right) \star f_{nl}(t), \quad (5.14)$$

where $\hat{k} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$, $\star$ denotes a temporal convolution, and

$$\int d^2 k = \int_0^{2\pi} d\phi \int_0^\pi d\sin \theta. \quad (5.15)$$

To relate $\tilde{A}_{nl}(r,t)$ to the vector potential $A_{nl}(r,t)$, the spectral integral in Eq. (5.14) is evaluated by interchanging the order of integrations and transforming the spatial variables to a new coordinate system in which $\hat{z}''$-axis is aligned with $R = r - r'$. Defining $\theta''$ and $\phi''$ as the angular coordinates in this new system, Eq. (5.14) reduces to [6]
\[ \tilde{A}_{n,l}(\mathbf{r}, t) = -\frac{\mu_0 c}{8\pi^2 c} \int_S d\mathbf{r}' \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' f_n(\mathbf{r}') f_{n,l}(t - (R \cos \theta')/c) \]

\[ = -\frac{\mu_0 c}{4\pi} \int_S d\mathbf{r}' f_n(\mathbf{r}') \frac{R}{R} \int_{-R/c}^{R/c} d\xi f_{n,l}(t - \xi) \]

\[ = \frac{\mu_0}{4\pi} \int_S d\mathbf{r}' f_n(\mathbf{r}') f_{n,l}(t - R/c) - \frac{\mu_0}{4\pi} \int_S d\mathbf{r}' f_n(\mathbf{r}') f_{n,l}(t + R/c). \]  

Comparing Eqs. (5.13) and (5.16), it is apparent that the evaluation of Eq. (5.14) yields the true vector potential (first term) and an anticausal signal (second term), which will henceforth be referred to as the ghost signal. As the ghost signal appears at the observer before the true signal arrives, it is possible to time-gate \( \tilde{A}_{n,l}(\mathbf{r}, t) \) and recover the true vector potential provided that certain conditions are met.

To develop a scheme for time-gating \( \tilde{A}_{n,l}(\mathbf{r}, t) \), the following observations are in order. With reference to Eq. (5.16), the true vector potential radiated by the subsignal \( f_{n,l}(t) \) reaches the observer no sooner than \( t = \mathbf{l}_I^{\text{min}} + \min\{R\}/c \), and the ghost signal vanishes after \( t = \mathbf{l}_I^{\text{max}} - \min\{R\}/c \). Hence, choosing the duration of the signal \( T_s < \min\{R\}/c \) implies that \( f_{n,l}(t) \) vanishes before the true signal reaches the observer and that the ghost and true fields never overlap in time. These observations can now be generalized for an arbitrary distribution of sources and observers residing in their respective spheres. For this configuration, the arrival of the true signal can be no sooner than \( (R_c - 2R_s)/c \) following the onset of a source signal at \( t = \mathbf{l}_I^{\text{min}} \). Hence, the choice \( T_s < (R_c - 2R_s)/c \) dictates that \( A_{n,I}(\mathbf{r}, t) = 0 \) for \( t < \mathbf{l}_I^{\text{max}} \) and \( A_{n,I}(\mathbf{r}, t) = \tilde{A}_{n,I}(\mathbf{r}, t) \) for \( t \geq \mathbf{l}_I^{\text{max}} \) as the ghost signal vanishes for \( t > \mathbf{l}_I^{\text{max}} - (R_c - 2R_s)/c \).

Summarizing these observations, the choice of \( T_s < (R_c - 2R_s)/c \) ensures that

\[ A_{n,I}(\mathbf{r}, t) = \begin{cases} 
0 & ; \ t < \mathbf{l}_I^{\text{max}} \\
\tilde{A}_{n,I}(\mathbf{r}, t) & ; \ t \geq \mathbf{l}_I^{\text{max}}.
\end{cases} \]  

(5.17)
It follows that both the electric and magnetic fields can be evaluated from Eq. (5.14). Indeed, using Eqs. (5.6) and (2.34), and the fact that \( \nabla \to -\partial_t \hat{\mathbf{k}}/c \) holds for plane wave basis, the fields in the observation sphere due to the \( l \)th source time slice for \( t \geq t_l^{\text{max}} \) can be written as

\[
\mathbf{E}_{n,l}(\mathbf{r}, t) = \frac{\eta_0}{8\pi^2 c^2} \int_{t_l^{\text{max}}}^{t} dt' \int d^2 \mathbf{k} (\mathbf{r} - \mathbf{r}'/c) \cdot \int_{S_n} dr' f_n(r') \delta \left(t' - \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')/c\right) \star f_{n,l}(t')
\]

\[
\mathbf{H}_{n,l}(\mathbf{r}, t) = \frac{1}{8\pi^2 c^2} \int_{t_l^{\text{max}}}^{t} dt' \int d^2 \mathbf{k} \times \int_{S_n} dr' f_n(r') \delta \left(t' - \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')/c\right) \star f_{n,l}(t').
\]

In deriving these equations, the fact that these fields should vanish for \( t < t_l^{\text{max}} \) due to Eq. (5.17) was implicitly incorporated in the temporal integration limits.

To explore the merits of a plane wave representation in constructing a fast algorithm, note that \( \tilde{\mathbf{A}}_{n,l}(\mathbf{r}, t) \) tested with a basis function \( f_m(\mathbf{r}) \) can be written as

\[
\langle f_m(\mathbf{r}), \tilde{\mathbf{A}}_{n,l}(\mathbf{r}, t) \rangle = -\frac{\mu_0}{8\pi^2 c} \int d^2 \mathbf{k} \left[ \int_{S_m} dr f_m(\mathbf{r}) \delta \left(t - \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_m)/c\right) \right]^{T} \star \delta \left(t - \mathbf{k} \cdot \mathbf{R}_c/c\right)
\]

\[
* \left[ \int_{S_n} dr' f_n(r') \delta \left(t + \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_n)/c\right) \right] \star f_{n,l}(t),
\]

where the superscript \( T \) is used to denote a transpose. Similarly, using Eqs. (5.3)-(5.5), (5.18) and (5.19), it can be shown that the inner products of the fields \( E_{n,l}(\mathbf{r}, t) \) and \( \mathbf{H}_{n,l}(\mathbf{r}, t) \) with \( f_m(\mathbf{r}) \) are zero for \( t < t_l^{\text{max}} \), and for \( t \geq t_l^{\text{max}} \) they can be expressed as

\[
\langle f_m(\mathbf{r}), \mathcal{E}_l \{ \mathbf{J}_{n,l}(\mathbf{r}, t) \} \rangle = \frac{\eta_0}{8\pi^2 c^2} \int_{t_l^{\text{max}}}^{t} dt' \int d^2 \mathbf{k} \left[ S_m^- (\mathbf{k}, t', \mathbf{k}) \right]^{T} \ast S_n^+ (\mathbf{k}, \mathbf{R}_c, t') \star f_{n,l}(t')
\]

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\[
\langle f_m(r), L_{\pi}(J_{n,l}(r,t)) \rangle = \frac{1}{8\pi^2c^2} \int_{t_{\text{max}}}^{t'} dt' \int d^2\hat{k} \left[ S_m^-(\hat{k},t',\hat{n}) \right]^T \cdot \mathcal{T}(\hat{k}, R_c, t') \cdot \left[ S_n^+(\hat{k},t',\hat{k}) \right] \cdot f_{n,l}(t'),
\]

and therefore
\[
\langle f_m(r), L_{\pi}(J_{n,l}(r,t)) \rangle = \frac{1}{8\pi^2c^2} \int_{t_{\text{max}}}^{t'} dt' \int d^2\hat{k} \left[ -\beta S_m^-(\hat{k},t',\hat{k}) + S_m^-(\hat{k},t',\hat{n}) \right]^T \cdot \mathcal{T}(\hat{k}, R_c, t') \cdot \left[ S_n^+(\hat{k},t',\hat{k}) \right] \cdot f_{n,l}(t').
\]

In the above equations, \( \mathcal{T}(\hat{k}, R_c, t) \) denotes the translation function
\[
\mathcal{T}(\hat{k}, R_c, t) = \partial_i \delta \left( t - \hat{k} \cdot R_c / c \right)
\]

and
\[
S_o^\pm(\hat{k}, t, \hat{\nu}) \equiv \int_{S_o} dr' \hat{\nu} \times f_o(r') \delta \left( t \pm \hat{k} \cdot (r' - r_o)/c \right),
\]

where \( o \in \{m,n\} \). Note that the nature of the projection operator \( S_o^+(\hat{k},t',\hat{k}) \) indicates that the scattered electric and magnetic fields can be constructed using the components of the vector potential that are transverse to the propagation direction \( \hat{k} \).

### 5.2.2 Practical implementation

Computing all interactions requires numerical implementation of Eq. (5.23). In practice, as all spherical integrals are computed using quadrature rules, it is necessary to temporally bandlimit the current densities. This follows naturally as the excitation is assumed to be bandlimited to \( \omega_{\text{max}} \). Thus, the current density can be locally interpolated using temporally
bandlimited and approximately time-limited functions such that the time signature of the current for the $l^{th}$ time interval can be written as

$$f_{n,l}(t) = \sum_{j=(l-1)M_t+1}^{lM_t} I_{n,j} \psi_j(t), \quad (5.26)$$

where $\psi_j(t) = \psi(t - j\Delta_t)$ is an interpolant bandlimited to $\omega_s > \omega_{max}$. In this study, $\psi_j(t)$ is chosen to be a variant of the approximate prolate spheroidal functions [104] (see Eq. (3.41) and Figure 5.2(b)). Time-limiting these functions to a duration of $(2p_t + 1)\Delta_t$ introduces an error which decreases exponentially with increasing $p_t$. As a result, $f_{n,l}(t)$ can be considered bandlimited and spans a duration $M_t'\Delta_t = (M_t + 2p_t)\Delta_t$. Evaluation of Eq. (5.23) is done by sampling the integrands on the surface of the sphere. As the far field is bandlimited, it is possible to show that the number of samples needed to completely characterize the field is $O(K^2)$, where

$$K = \lceil 2\chi \omega_s R_s / c \rceil + 1 \quad (5.27)$$

and $\chi$ is an oversampling factor [98, 99]. Note that the translation function in Eq. (5.24) can be succinctly expressed as $T(\hat{k}, R_c, t) = \tilde{T}(\hat{k}, R_c, t, \infty)$, where [85]

$$\tilde{T}(\hat{k}, R_c, t, K') = \frac{c\varphi^3}{2R_c} \sum_{\nu=0}^{K'} (2\nu + 1)P_{\nu}(ct/R_c)P_{\nu}(\cos \theta^*) \quad \text{for} \quad |t| \leq R_c / c \quad (5.28)$$

and $P_{\nu}(-)$ denotes the Legendre polynomial of degree $\nu$. In keeping with the bandlimitedness of the fields, the upper limit in the above summation can be truncated to $K' = K$. Consequently, Eq. (5.23) can be evaluated numerically using

$$\begin{equation}
\langle f_{m}(r), L_c\{J_{n,l}(r, t)\} \rangle = \frac{1}{8\pi c^2} \int_{l_{max}}^{l} dt' \sum_{k=0}^{K} \sum_{p=-K}^{K} w_{kp} \left[ -\beta S_m^-(\hat{k}_{kp}, t', \hat{k}_{kp}) + S_m^+(\hat{k}_{kp}, t', \hat{n}) \right]^T \ast \tilde{T}(\hat{k}_{kp}, R_c, t', K) \ast \left[ S_n^+(\hat{k}_{kp}, t', \hat{k}_{kp}) \right] \ast f_{n,l}(t') \tag{5.29}
\end{equation}$$
where $w_{kp}$ and $\hat{k}_{kp}$ are, respectively, the quadrature weights and direction samples given by

$$
\begin{align*}
  w_{kp} &= \frac{4\pi(1 - \cos^2 \theta_k)}{(2K + 1)[(K + 1)P_K(\cos \theta_k)]^2}, \\
  \hat{k}_{kp} &= \hat{x} \sin \theta_k \cos \phi_p + \hat{y} \sin \theta_k \sin \phi_p + \hat{z} \cos \theta_k, \\
  \phi_p &= p2\pi/(2K + 1), \\
  \theta_k &= \text{the (k + 1)th zero of } P_{K+1}(\cos \theta).
\end{align*}
$$

(5.30)

To gain insight into Eq. (5.29), note that the rightmost convolution maps the source distribution onto a set of plane waves, which will henceforth be referred to as “outgoing rays.” In the literature, this mapping is also known as the slant stack transform (SST) [6]. The center convolution “translates” these to “incoming rays,” which impinge on the observation sphere. Finally, via the last convolution and the spectral integration the incoming rays are mapped onto the observers. The reconstruction of the transient field using this three-stage process of aggregation, translation, and disaggregation is reminiscent of the popular frequency domain FMM.

### 5.3 The Two-Level PWTD Enhanced MOT Algorithm

To this end, assume that the scatterer can be enclosed in a fictitious cubical box, which is further subdivided in many smaller equal-sized cubes or boxes (Figure 3.8(a)). The radius of a sphere circumscribing each cube is denoted by $R_s$, and the set of basis functions that are contained in a box is called a group. Then, all nonempty boxes are identified and numbered as $\gamma = 1, \ldots, N_g$. Next, a pair of boxes $(\gamma, \gamma')$ is classified as either a “near-field” or a “far-field” pair, depending on a separation criterion based on the distance between the box centers. In our analysis, this distance was chosen to be $R_{c,min} < \xi R_s$, where $4 \leq \xi \leq 6$. Since $R_{c,min} = O(R_s)$, the number of near-field pairs is proportional to the total number of nonempty boxes $N_g$. The
interactions between the basis functions that reside in a near-field box pair are computed using the classical MOT scheme. However, the interactions between those that reside in a far-field box pair are computed in a group-wise manner using the PWTD algorithm. The foundations of the PWTD algorithm lie in expressing the field at a point due to a sufficiently separated source distribution as a superposition of plane waves. Such a representation, as in the FMM, has been extremely effective in reducing the computational complexity of the frequency domain integral equation solvers. Consequently, the analysis in the first part of this section will focus on the development of such a scheme in the time domain. Next, the incorporation of this algorithm into an MOT scheme in a two-level setting is elucidated. Finally, it is shown that the resulting PWTD-enhanced MOT algorithm has a computational complexity of $O(N_tN_s^{1.5} \log N_s)$.

5.3.1 The algorithm and its computational complexity

The ideas outlined in the preceding subsection give rise to a procedure via which the interaction between basis functions residing in any far-field box pair $(\gamma, \gamma')$ can be computed as a superposition of transient plane waves. To complete the PWTD algorithmic prescription, these ideas are now systematized such that they can be efficiently applied in conjunction with an MOT scheme. With this in mind, a fundamental subsignal duration $T_s = (M_t + 2p_t)\Delta t$ is defined, where

$$M_t = \min_{(\gamma,\gamma')} \left\{ \left[ \frac{R_{c,\gamma\gamma'} - 2R_s}{c\Delta t} \right] - 2p_t \right\}$$  \hspace{1cm} (5.31)

and $R_{c,\gamma\gamma'}$ is the distance between the centers of the boxes $(\gamma,\gamma')$. Such a definition stems from the fact that $T_s$ corresponds to the duration of the longest possible subsignal that can be translated “ghost-free” between the closest far-field pair. For all other far-field pairs, the subsignal lengths are chosen to be an integer multiple of the fundamental duration, i.e.,
\[ T_{\gamma,\gamma'} = (M_t,\gamma') + 2p_t \Delta t \quad \text{where} \quad M_t,\gamma' = M_t \left[ (R_c,\gamma' - 2R_s - 2p_t c \Delta t) / M_t \right]. \] The rationale behind this choice will soon become transparent.

The task of computing the current distribution at each time step is divided into (i) evaluating near-field interactions using the usual MOT scheme, and (ii) computing far-field interactions using the PWTD algorithm.

1. **Near-Field Evaluation**: At each time step, the sum

\[
\sum_{i=1}^{j-1} \sum_{n \in \gamma'} \sum_{n \in \gamma'} Z_{q,mn} I_{j-i,n} \quad \forall m \in \gamma
\]

(5.32)

is computed for all near-field interaction pairs \((\gamma', \gamma')\).

2. **Far-Field Evaluation**: To take all the far-field interactions into account, the algorithm follows the three-stage process that was alluded to in the previous subsection.

   a) The first task is the construction of outgoing rays for all boxes. This involves computing \( S^*_n(k_{kp}, t, k_{kp}) \) for all ray directions for a fundamental subsignal comprised of \( M_t + 2p_t \) samples and duration \( T_s \). Note that, once this information has been computed, it can be reused for different interaction pairs. Furthermore, the number of time slices for which one needs to store these outgoing rays is proportional to the largest linear dimension of the scatterer.

   b) The next step is to translate the outgoing rays from a source group \( \gamma' \) to an observer group \( \gamma \). This is done every \( M_t,\gamma' \) time steps. As \( M_t,\gamma' \) is an integer multiple of \( M_t \), the rays to be translated can be formed by concatenating an appropriate number of outgoing rays from group \( \gamma' \). Since the length of the translation function is \( 2R_c,\gamma' / (c \Delta t) = O(M_t,\gamma') \) time steps, which is of the same order as the length of the
outgoing ray to be translated, the convolution of the outgoing ray with the translation function can be efficiently accomplished using fast Fourier transforms (FFTs). Unfortunately, as the translation function is not bandlimited, simple FFTs cannot be used to transform it into the frequency domain. However, this hurdle can be surmounted as an analytical expression for the Fourier transform of the translation function is available. For a far-field pair \((\gamma, \gamma')\), the Fourier transform of the translation function is

\[
\mathcal{F}\{\mathcal{T} (\hat{k}, R_{c,\gamma\gamma'}, t, K')\} = \int_{-\infty}^{\infty} dt \, e^{-j\omega t} \mathcal{T} (\hat{k}, R_{c,\gamma\gamma'}, t, K') = \int_{-\infty}^{\infty} dt \, e^{-j\Omega ct/R_{c,\gamma\gamma'}} \mathcal{T} (\hat{k}, R_{c,\gamma\gamma'}, t, K')
\]

\[
= \frac{c^2}{R_{c,\gamma\gamma'}} (j\Omega)^3 \sum_{\nu = 0}^{K' \prime} (2\nu + 1)(-j)^\nu j_\nu(\Omega) P_\nu(\cos \theta^*)
\]

\[
= \frac{c^2}{R_{c,\gamma\gamma'}} \mathcal{T} (\hat{k}, \Omega, K') \tag{5.33}
\]

where \(\Omega = \omega R_{c,\gamma\gamma'}/c\) is the normalized frequency, and \(j_\nu(\cdot)\) is a spherical Bessel function of order \(\nu\) [85]. This equation also indicates that the translation function for an arbitrary far-field pair can be constructed from the function \(\mathcal{T} (\hat{k}, \Omega, K')\) that is bandlimited in both \(\Omega\) and \(\theta\). Hence, this function can be sampled at a discrete set of points, and the translation function for any far-field pair can be reconstructed by interpolating through these samples. After convolving the outgoing rays with the translation function, the resulting rays are then superimposed onto the incoming rays of the observer group. It should be noted that as the evaluation of \(S_n^+ (\hat{k}_{kp}, t, \hat{k}_{kp})\) yields two real signals, viz., the \(\theta\) and \(\phi\) components of the field, which are to be translated, this operation is most efficiently performed using one complex FFT [105].

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c) Finally, the rays entering all the spheres are projected on to the observers. This is done via the leftmost convolution and the double sum in Eq. (5.29).

It should be pointed out that the error incurred in computing the fields via a plane wave expansion method can be controlled to arbitrary precision (see Chapter 3).

To analyze the computational complexity of the two-level PWTD-enhanced MOT solver described above, assume that there are $N_g$ nonempty boxes, each containing approximately $M_s = N_s / N_g$ unknowns. In what follows, implementation-dependent constants are denoted as $C_{iT}$, $i = 0, 1, 2, 3$. The cost of a PWTD-MOT analysis is comprised of near- and far-field components. The number of operations required for the computation of all near-field interactions for the duration of the analysis asymptotically scales as

$$C_{NF} = C_0 N_g M_s^2 N_t. \quad (5.34)$$

operations. The cost of evaluating all far-field interactions consists of those incurred in constructing outgoing rays ($C_{1FF}^1$), translating the latter onto incoming rays ($C_{1FF}^2$), and projecting incoming rays onto observers ($C_{1FF}^3$). Constructing outgoing rays involves the projection of all current elements in the $N_g$ nonempty boxes onto $O(K^2) = O(M_s)$ plane waves for all $N_t$ time steps. Projecting incoming rays onto observers involves a very similar set of operations. Hence, the cost associated with these operations asymptotically scales as

$$C_{NF}^{1,3} = C_{1,3} N_g M_s^2 N_t \quad (5.35)$$

The cost of translating one ray between groups ($\gamma, \gamma'$) scales as $O(M_{1,\gamma', \log M_{1,\gamma'}})$. As this operation has to be performed for all $N_t / M_{1,\gamma'}$ time intervals and $O(K^2) = O(M_s)$ directions, the cost of translating information between a given pair of boxes for the duration of the analysis scales as $O(N_t M_s \log M_{1,\gamma'})$. Since $M_{1,\gamma'}$ is bounded by the maximum linear dimension of the scatterer ($\sqrt{N_s}$), the cost of translations for all $N_g^2$ far-field group interactions scales as
\[ C_{FF}^2 = C_2 N_1 N_s^2 M_s^{-1} \log N_s. \]  \hspace{1cm} (5.36)

The total cost associated with the PWTD-enhanced MOT analysis is

\[ C_T = C_{NF} + C_{IF}^1 + C_{IF}^2 + C_{IF}^3. \]  \hspace{1cm} (5.37)

Minimizing \( C_T \) with respect to \( M_s \) reveals that the optimal number of unknowns per group scales as \( M_s \propto \sqrt{N_s} \), and that \( C_T \) scales as \( O(N_1 N_s^{1.5} \log N_s) \).

### 5.3.2 Numerical results

This section presents a collection of numerical results that serve both to validate the above-described PWTD-enhanced MOT scheme and to demonstrate its efficacy. The algorithm is verified in two stages. In a first set of experiments, scattering from electrically small structures is analyzed using PWTD-enhanced and classical EFIE- and MFIIE-based MOT codes, and compared in the time domain directly. In a second set of experiments, frequency domain radar cross section (RCS) data is extracted from the Fourier transformed temporal far-field signatures [4] obtained using a PWTD-augmented CFIE-based MOT solver. This data is compared against similar results obtained using the Fast Illinois Solver Code (FISC)—an FMM-based CFIE frequency domain code [53]. In all the examples presented herein, the incident pulse is a modulated Gaussian plane wave parameterized as

\[ E^i(r,t) = \hat{p} \cos(2\pi f_o (t - r \cdot \hat{k}/c)) \exp \left[ -\frac{(t - r \cdot \hat{k}/c - t_p)^2}{2\sigma^2} \right], \]  \hspace{1cm} (5.38)

where \( f_o \) is the pulse’s center frequency, \( \hat{k} \) and \( \hat{p} \) denote its direction of travel and polarization, \( \sigma = 6/(2\pi f_{bw}) \), and \( t_p = 3.5\sigma \). The parameter \( f_{bw} \) will be referred to as the “bandwidth” of the signal. It is to be noted that the power in the incident pulse is down by 160 dB at \( f = f_o \pm f_{bw} \) relative to that at \( f_o \).
To establish that both PWTD-enhanced and classical MOT schemes yield identical results, transient scattering from two objects is analyzed. The first object is a rectangular plate of dimensions $2 \times 5$ m that resides in the $xy$-plane as shown in the inset of Figure 5.3(a). A $\hat{p} = \hat{x}$ polarized pulse traveling in the $\hat{k} = -\hat{z}$ direction, with center frequency $f_o = 100$ MHz and bandwidth $f_{bw} = 50$ MHz is incident on the plate. The current on the plate is represented using 2170 spatial basis functions and solved for using the EFIE. The magnitudes of the current at $(0.2, 2.0, 0.0)$ on the plate’s surface, computed using both the PWTD-enhanced and classical MOT schemes, are compared in Figure 5.3(a). Similarly, the temporal far-field signatures of the field scattering along $+\hat{z}$ direction, computed using both methods, are compared in Figure 5.3(b). Obviously, the PWTD-enhanced MOT solver yields results that agree very well with those obtained using the classical scheme.

![Figure 5.3: Transient scattering from a plate analyzed using the EFIE. The plate is discretized with 2170 unknowns. The plate measures $2 \times 15$ m. The incident field is $\hat{k} = -\hat{z}$ directed and $\hat{p} = \hat{x}$ polarized. (a) Current at a location on the plate. (b) The backscattered far field $E_x$.](image)

The second object studied is an almond that fits into a rectangular box of size $5.00 \times 1.92 \times 0.64$ m. The almond is excited by a pulse traveling along $\hat{k} = -\hat{z}$ and polarized
along $\hat{p} = \hat{x}$. The center frequency and bandwidth of the pulse are $f_0 = 204$ MHz and $f_{bw} = 200$ MHz, respectively. The current on the almond is represented using 4680 spatial basis functions and solved for using an MFIE. Figure 5.4 compares the magnitude of the current at the location $(0.01, 0.58, 0.03)$ on the almond and the far field signature of the $+\hat{z}$ traveling scattered field obtained using PWTD-enhanced and classical MOT codes; again, good agreement between both sets of results is observed.

![Graphs showing current and field magnitude](image)

(a) (b)

Figure 5.4: Transient scattering from an almond, which is discretized using 4680 spatial basis functions, analyzed using the MFIE. The incident field propagates along $\hat{k} = -\hat{z}$ and is $\hat{p} = \hat{x}$ polarized. (a) Current at a location on the almond. (b) The backscattered far field $E_x$.

Via the above experiments, it has been verified that the solutions to the EFIE and the MFIE obtained using PWTD-enhanced and classical MOT schemes are in agreement. Next, the PWTD-accelerated MOT code is applied to the analysis of scattering from electrically large objects, and is further validated by comparing their bistatic RCS—extracted at a number of frequencies—against RCS data computed using a frequency domain solver. In all the examples that follow, the time domain CFIE is used as the solution to the time domain EFIE, and MFIE can be corrupted by internal resonance modes. In a first experiment, transient scattering from a
sphere of unit radius is analyzed. The sphere is illuminated by a modulated Gaussian pulse traveling in the \( \hat{\mathbf{k}} = -\hat{\mathbf{z}} \) direction and polarized along \( \hat{\mathbf{p}} = \hat{\mathbf{x}} \). The center frequency and bandwidth of the pulse are \( f_o = 204 \) MHz and \( f_{bw} = 200 \) MHz, respectively, and the sphere is discretized using 9414 spatial basis functions. Figure 5.5 compares the RCS pattern in the \( xz \)-plane obtained using the PWTD-enhanced MOT scheme and FISC for a range of frequencies within the band of excitation. Specifically, the RCS is compared at 280, 350, 420 and 500 MHz. Results obtained from the time and frequency domain codes are in good agreement with each other, even at 280 and 500 MHz, where the power in the incident pulse is down by 65 and 40 dB from its peak at \( f_o \), respectively. In this and subsequent examples, comparisons at frequencies where the power is down by at least 30 dB are made to highlight the fact that meaningful results can be obtained at these points through the use of the CFIE. Indeed, if either the MFIE or the EFIE were used, errors induced by nonphysical currents at resonance frequencies would propagate in any MOT scheme and would be most visible at the ends of the band.

In a second experiment, transient scattering of a pulse traveling along \( \hat{\mathbf{k}} = -\hat{\mathbf{z}} \) and polarized along \( \hat{\mathbf{p}} = \hat{\mathbf{x}} \) from an almond that fits into a rectangular box of dimensions \( 5.0 \times 1.92 \times 0.64 \) m is analyzed again. However, this time, the center frequency and bandwidth of the pulse are \( f_o = 303.4 \) MHz and \( f_{bw} = 200 \) MHz, respectively, and the almond is discretized using 10620 spatial basis functions. RCS patterns in the \( xz \)-plane are extracted from temporal far field signatures at 210.2, 261.8, 322.0, and 390.8 MHz, and compared against FISC data in Figure 5.6; again, all RCS patterns agree well with each other. It should also be pointed out that at both 210.2 and 390.8 MHz the power in the incident pulse is down by about 30 dB from its peak at \( f_o \).
Figure 5.5: The $xz$-plane RCS pattern of a sphere extracted from the time domain CFIE result is compared to that obtained from FISC for four different frequencies. The incident wave propagates along $\hat{k} = -\hat{z}$ and is $\hat{p} = \hat{x}$ polarized, and the sphere is discretized using 9414 spatial basis functions.
Figure 5.6: The $xz$-plane RCS patterns of an almond extracted from the time domain CFIE result are compared to those obtained from FISC at four different frequencies. The incident wave propagates along $\hat{k} = -\hat{z}$ and is $\hat{p} = \hat{x}$ polarized, and the almond is discretized using 10620 spatial basis functions.

Next, scattering from a cone-sphere is studied. The cone is 1 m long and the radius of the half sphere attached to the cone is 0.235 m. The incident pulse has a center frequency of $f_0 = 800$ MHz and bandwidth of $f_{bw} = 750$ MHz, is $\hat{p} = \hat{z}$ polarized, and travels along the $\hat{k} = -\hat{y}$ direction. This cone-sphere is discretized using 11412 spatial basis functions. The RCS
patterns in the yz-plane obtained from the time domain PWTD-enhanced MOT code are compared against those obtained from FISC at 500, 700, 900, 1100 MHz, as shown in Figure 5.7. Examination of these figures shows that both the FISC and the PWTD results agree very well with each other, even at 500 and 1100 MHz where the power in the incident pulse is down by 25 dB from that at 750 MHz.

Figure 5.7: The yz-plane RCS patterns of a cone-sphere extracted from the time domain CFIE result are compared to those obtained from FISC at four different frequencies. The incident wave propagates along $\hat{k} = -\hat{y}$ and is $\hat{p} = \hat{z}$ polarized, and the cone-sphere is discretized using 11412 spatial basis functions.
All examples analyzed thus far involved relatively simple geometries. Next, scattering from a VFY 218 aircraft, discretized using 9747 spatial basis functions, is analyzed. The incident field travels in the $\hat{k} = -\hat{y}$ direction and is polarized along $\hat{p} = \hat{x}$, has a center frequency of $f_o = 100$ MHz, and bandwidth of $f_{bw} = 60$ MHz. Figure 5.8 compares the RCS patterns in the $xy$-plane at 75, 85, 105, and 130 MHz, where the pulse's power at these frequencies is down by 27, 9.7, 1.07, and 39 dB with respect to its peak value. Again, the results of the time domain code replicate all of the RCS nulls and peaks computed using the frequency domain solver.

Next, scattering from an almond discretized with 29700 spatial basis functions is analyzed. This experiment pushes the limits of the two-level PWTD algorithm insofar as computational resources are concerned. This almond fits inside a rectangular box of dimensions $6.0 \times 2.31 \times 0.77$ m, and is illuminated by a $\hat{p} = \hat{x}$ polarized Gaussian pulse traveling along $\hat{k} = -\hat{z}$, with a center frequency of $f_o = 500$ MHz and a bandwidth of $f_{bw} = 300$ MHz. Figure 5.9 compares the RCS patterns in the $xz$-plane computed using the time and frequency domain codes at 400, 460, 520, and 600 MHz. At both 400 and 600 MHz, power in the incident field is down by 27 dB from its peak. As is apparent in Figure 5.9, the RCS patterns computed by both the time domain and frequency domain codes are in very good agreement with each other.

Finally, the predicted computational complexity of the PWTD-enhanced MOT schemes is verified. All results presented above were obtained using an SGI Origin2000 with peak performance rated at 360 Mflops. In Figure 5.10, the logarithm of the CPU time required to compute the interactions at one time step is plotted against the logarithm of $N_s$. This graph reveals that the computational cost of a PWTD-MOT algorithm scales as $O(N_t N_s^{1.507} \log N_s)$, which is very close to the theoretically predicted scaling law. Also, it should be noted that the
break-even point, or the number of unknowns where it becomes more advantageous to use the PWTD-enhanced MOT schemes as opposed to a classical MOT solver, is as low as $N_x = 700$.

Figure 5.8: The xy-plane RCS patterns of an aircraft (VFY 218) extracted from the time domain CFIE result are compared to those obtained from FISC at four different frequencies. The incident wave propagates along $\hat{k} = -\hat{y}$ and is $\hat{p} = \hat{x}$ polarized, and the cone-sphere is discretized using 9747 spatial basis functions.
Figure 5.9: The xz-plane RCS patterns of an almond extracted from the time domain CFIE result are compared to those obtained from FISC at four different frequencies. The incident wave propagates along $\hat{k} = -\hat{z}$ and is $\hat{p} = \hat{x}$ polarized, and the almond is discretized using 29700 spatial basis functions.
Figure 5.10: Comparison of the computational complexity of the classical and PWTD augmented MOT schemes.

5.4 The Multilevel PWTD Enhanced MOT Algorithm

The previous section showed that the computational complexity of constructing the right-hand side of Eq. (5.7) can be reduced to $O(N_tN_s^{1.5}\log N_s)$ from $O(N_tN_s^2)$ using the PWTD algorithm in a two-level setting. Next, the ideas presented in the previous section will be generalized such that the PWTD algorithm can be utilized within a multilevel setting. To this end, in Section 5.4.1, first the multilevel notation introduced in Section 4.4.1 will be reviewed. Then, the components of multilevel scheme will be elucidated and computational requirements of each component estimated. As the present algorithm is very similar in spirit to that developed for the acoustic case, this section will be kept brief. Finally, in Section 5.4.2, the capabilities of the multilevel scheme will be explored through several numerical examples.
5.4.1 The algorithm and its computational requirements

To systematically describe the multilevel scheme, consider a cubical box that encloses the scatterer. The first step in implementing an $N_l$ level scheme is to recursively subdivide this cubical box associated with level $N_l$ into finer boxes. Hence, starting at level $N_l - 1$, at each level, each parent box of the previous level is subdivided into eight equal-sized child boxes. The radius of the circumscribing sphere for all level $i$ boxes is denoted by $R_s(i)$ and a fundamental subsignal duration $T_s(i) \propto R_s(i)/c$ is chosen. This process is continued until level 1, where $R_s(1)$ is a fraction of the smallest wavelength of interest. During this recursive subdivision, nonempty boxes at all levels are identified. Next, the far-field interaction pairs at a level $i$ are identified and tabulated using the following argument: two boxes $(\gamma, \gamma')$ are said to be in each other's far-field if the distance between their centers is greater than a prescribed distance $R_{min}(i) \geq 2R_s(i) + cT_s(i)$ and if their parents are in each other's near-field. At the lowest level, all box pairs which are separated by a distance less than $R_{min}(1)$ are classified as near-field pairs. These arguments are illustrated in Figure 3.8(b).

Before embarking on an algorithmic description of the multilevel PWTD, the following observations are in order: At any level $i$, the fundamental segment duration $T_s(i) = 2T_s(i - 1)$, and $R_s(i) = 2R_s(i - 1)$. Also, equivalents of all equations in Section 5.2 that are valid at any level $i$ may be obtained by replacing $R_s$ and $T_s$ with $R_s(i)$ and $T_s(i)$. With this background, the multilevel PWTD algorithm that complements the MOT solver is now described in two parts: (i) near-field evaluation and (ii) far-field evaluation.

1. Near-Field Evaluation: At each time step, the interaction between all near-field pairs is computed using the classical MOT algorithm. As there are $O(1)$ unknowns per box at the lowest level and $O(N_s)$ boxes, this operation scales as $O(N_s)$ per time step, and requires a
total of $O(N_tN_s)$ operations for the duration of the analysis. Similarly, since each box interacts with a finite number of boxes, the memory required to store these interactions scales as $O(N_s)$. Implicit in arriving at this storage estimate is the fact that to compute the interaction between two basis functions, one needs to store a finite set of numbers which depend on the time step size and the order of the temporal interpolation function.

2. **Far-Field Evaluation**: The interaction between far field groups is evaluated using the three-stage PWTD algorithm, in which each stage corresponds to the evaluation of a convolution appearing in Eq. (5.29). However, in the multilevel scheme, in order to establish the flow of information between the levels, two more sets of operations, interpolation and splicing, and resectioning and anterpolation complement the first and the last convolutions as follows.

   a) **Construct Outgoing Rays**: The first stage is the computation of outgoing rays for all boxes. This is done by first computing $S_n^+(\k_{kp}, t, \k_{kp})*f_{n,i}(t)$ for all directions $\k_{kp}$ and for all boxes at the lowest level, with each ray spanning a duration of $O(T_s(1))$. The total cost associated with generating the outgoing rays at the lowest level scales as $O(N_tN_s)$. These rays are constructed at the end of each time segment. Once the rays for the lowest level boxes have been computed, rays for all higher level boxes are constructed using interpolation and splicing introduced in Section 3.4.2. For instance, to compute the contribution of the box $\gamma_c$ to the rays of its parent $\gamma_p$, the first step is to interpolate the rays of the child box to a larger number of directions $\k_{kp}'$ as dictated by its parent. Then each ray is shifted by an amount $\k_{kp}' \cdot R_{c,\gamma_c\gamma_p}$ and superimposed onto a ray of the parent box. Note that as $T(i) = 2T(i - 1)$, two consecutive rays of a child box contribute to one ray of its parent box. The process of combining two consecutive rays from each child box to form one ray of the parent is called splicing. This procedure is repeated for all boxes at all levels. The cost of forming outgoing rays at a level by
interpolation and sectioning is proportional to the number of groups times the cost of one interpolation times the number of times interpolation is done throughout the analysis. Denoting the number of unknowns that reside in a box at level \( i \) by \( M_s(i) \), the cost for constructing outgoing rays for a box at level \( i + 1 \) scales as \( O(M_s(i) \log M_s(i)) \) per time step if the interpolation algorithm that is described in [102] is used. Hence, the total cost of forming outgoing rays at level \( i + 1 \) is \( O(N_s/M_s(i) \times M_s(i) \log M_s(i) \times N_t) \). This implies that the computational expense incurred in constructing outgoing rays for all levels is bounded by a cost that scales as \( O(N_t N_s \log^2 N_s) \). Likewise, the memory required to store the outgoing rays for level \( i \) is proportional to the total number of groups times the number of ray directions for a group times the length of each ray and is of \( O(N_s/M_s(i) \times M_s(i) \times T_s(i)/\Delta_t) \). Noting that \( T_s(N_t) = N_t \Delta_t \), it can be shown that the total memory required to store the outgoing rays scales as \( \sum_i O(N_s T_s(i)/\Delta_t) = O(N_t N_s) \). In arriving at this estimate, it is implicitly assumed that outgoing rays at any level need to be stored for a small set of time segments. It can be easily verified that number of time segments that is stored corresponds to \( \max_{\gamma, \gamma'} \{(R_c, \gamma' - 2R_s(i))/cT_i\} \).

b) Translation: The next step is to translate outgoing rays between far-field pairs at all levels, i.e., to form incoming rays. The outgoing rays of the pair \((\gamma, \gamma')\) are translated every \( T_{s, \gamma}' \) s, where \( T_{s, \gamma}' = T_s(i) \left( (R_c, \gamma' - 2R_s(i))/(cT_s(i)) \right) \). This choice of \( T_{s, \gamma}' \) dictates that the ray to be translated can be formed by concatenating an integer number of outgoing rays at that level. Since the duration of the translation function is also \( O(T_{s, \gamma}, \gamma') \), the convolution is most efficiently accomplished using the fast Fourier transform (FFT). The total cost of this operation for all levels scales as \( O(N_t N_s \log^2 N_s) \). However, this complexity is contingent upon the fact that the Fourier transform of the translation function for the pair \((\gamma, \gamma')\) be either readily available or that
it can be constructed relatively cheaply. As in the two-level scheme, the Fourier transformed translation function between any box pair at a level can be computed by first tabulating the function \( \hat{T}(\hat{k}, \Omega, K) \) given in Eq. (5.33) at a discrete set of angles and frequencies at a preprocessing stage, and then using interpolation to construct the translation function whenever necessary. This entails \( O(\sqrt{M_S(i)}) \) operations for evaluating the translation function per direction for a box pair at level \( i \). Thus, the overall cost of computing the translation function scales as \( O(N_t N_S) \). However, while the asymptotic complexity of this operation is quite low, the break-even point, compared to the direct evaluation of Eq. (5.33), is reached only when the scatter is large enough to use a four-level algorithm. Alternatively, since the number of unique \( R_{c, \gamma'} \) at any level is very small, one can precompute and store the translation function for \( O(R_{c, \gamma'} / c \Delta_t) \) frequency samples and for sampled set of angles. When desired, the translation function can be reconstructed using interpolation only in the angular direction. This technique has a lower breakeven point (for a three-level algorithm) albeit with a slightly higher memory requirement.

c) Construct Responses at Observers: Finally, the information that is contained in the incoming rays is used to construct responses at the observers. Starting from level \( N_1 \), the incoming rays of all parent boxes are resected and projected on to the incoming rays of their children via anterpolation. This process proceeds until the finest level, when the incoming rays at level 1 are convolved with

\[
\begin{bmatrix}
-\beta S_m(\hat{k}_{kp}, t, \hat{k}_{kp}) + S_m(\hat{k}_{kp}, t, \hat{n})
\end{bmatrix}^T
\]

in order to map the responses onto the observers. The resectioning operation is a dual of the splicing operation that was used to construct outgoing rays. To construct a simple resectioning scheme, the following observation is in order: For any box \( \gamma_p \) at level \( i \), the minimum time duration that can possibly elapse between any two successive sets of translations is \( T_s(i) \). This implies that the information contained in the segment

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\(2R_s(i)/c\) inside the sphere and \(T_s(i)\) outside the sphere is complete, i.e., later translations will not contribute to the information that exists in this segment. Hence, this segment of the incoming ray can be safely resected to form the incoming ray of the child box \(\gamma_c\) at level \(i-1\) with a shift of \(\hat{k}_{kp}\cdot R_{c,\gamma_c\gamma_p}\). The rays that are resected are then interpolated to the directions corresponding to level \(i-1\). Interpolation is a conceptual transpose of interpolation and is best understood by studying the above described operation from a slightly different perspective. It is well known that the spectral content of the field that is received by a set of observers is proportional to the size of the box enclosing it [98]. Via the process of interpolation, the fields received by the child boxes are constructed using the fields that impinge on their parent box. Consequently, this operation is tantamount to reducing the spatial spectral content of the fields on the larger box such that it can be used to represent the fields in the smaller box. In most implementations of the fast multipole method, interpolation is usually done using a transpose of the interpolation matrix. However, this method does not ensure that the higher order spatial harmonics are discarded. Consequently, errors are incurred as one traverses down the tree. In the multilevel PWTD scheme, interpolation is done using the spherical filter that is outlined in [102]. Using arguments similar to those used to find the asymptotic computational complexity for constructing the outgoing rays, it can be shown that the cost for this operation is bounded from above by \(O(N_tN_s\log^2 N_s)\). Likewise, the memory required to store the incoming rays also scales as \(O(N_tN_s)\).

As the computational complexity of all the three stages of the far-field computations is of \(O(N_tN_s\log^2 N_s)\), which is higher than the \(O(N_tN_s)\) complexity of the near-field part, the overall cost of the algorithm scales as \(O(N_tN_s\log^2 N_s)\). On the other hand, the memory for near- and far-field storage scale as \(O(N_s)\) and \(O(N_tN_s)\), respectively.
5.4.2 Numerical results

The objective of this section is two-fold: (i) to demonstrate that the error incurred in using the multilevel PWTD (ML-PWTD) algorithm is controllable to arbitrary precision, and (ii) to demonstrate the usefulness of this algorithm in analyzing scattering from electrically large structures.

In any computational scheme, it is desirable that the error incurred in using that methodology be controllable to arbitrary precision and that an a priori estimate of the error can be obtained. A set of experiments was conducted to determine such an error estimate. While a similar exercise was undertaken when establishing the validity of the two-level PWTD scheme, it should be noted that the multilevel scheme comes with two additional sets of operations; interpolation and splicing, and resection and anterpolation. The numerical experiment to determine controllability of the error of the ML-PWTD scheme is set up as follows: A set of sources is located at the vertices of a cube of dimension 0.5 m$^3$. Each source is assumed to be a $\hat{y}$ directed dipole and radiates a Gaussian pulse of the form $\exp[-(t-t_0)^2/(2\sigma^2)]$, which is parameterized by the constants $\sigma = 3.183 \times 10^{-9}$ s, and $t_0 = 2.4669 \times 10^{-8}$ s. The vector potential due to these sources is computed at a set of observers that are located in a cube of dimension 4.5 m$^3$ that is centered at (14.75, 14.75, 14.75) in meters. The entire system is then enclosed in a fictitious cubical box of dimension 18 m$^3$ (see Figure 5.11). Three sets of experiments are run with the $N_l$ ranging from 3 to 5. The location of the sources is such that they are fully contained in one lowest-level box. On the other hand, for any given $N_l$, different sets of observers may interact with the sources at different levels. The levels at which these interactions are computed are termed active levels. With such a geometrical configuration, three sets of results are obtained by varying $N_l$. The $L_2$ error in the observed field is obtained by comparing the numerical results thus obtained to those computed analytically. Table 5.1 lists the
number of levels that are active, the estimated error, and the average error in computation. As is evident, both the average error in the observation domain and the estimated error are of the same order of magnitude. This indicates that despite traversing up and down the levels, we have an \textit{a priori} estimate of the error incurred in using the PWTD algorithm, and the overall error can be controlled to the desired precision.

![Observer Box]

![Source Box]

Figure 5.11: Geometrical configuration that is used to compute the error characteristics of the multilevel PWTD algorithm.

Table 5.1: Comparison of the expected error and the average of the $L_2$ error in the observed fields.

<table>
<thead>
<tr>
<th>Active Levels</th>
<th>Estimated Error</th>
<th>Average Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$3.0 \times 10^{-3}$ $2.4 \times 10^{-6}$ $9.8 \times 10^{-10}$</td>
<td>$3.4 \times 10^{-3}$ $1.4 \times 10^{-6}$ $1.4 \times 10^{-9}$</td>
</tr>
<tr>
<td>3, 4</td>
<td>$3.0 \times 10^{-3}$ $2.4 \times 10^{-6}$ $9.8 \times 10^{-10}$</td>
<td>$5.1 \times 10^{-3}$ $2.1 \times 10^{-6}$ $2.1 \times 10^{-9}$</td>
</tr>
<tr>
<td>3, 4, 5</td>
<td>$3.0 \times 10^{-3}$ $2.4 \times 10^{-6}$ $9.8 \times 10^{-10}$</td>
<td>$7.0 \times 10^{-3}$ $2.8 \times 10^{-6}$ $4.2 \times 10^{-9}$</td>
</tr>
</tbody>
</table>
Having established the accuracy of the multilevel algorithm, the next task is to demonstrate its efficacy in analyzing transient scattering from electrically large structures via a series of numerical experiments. These experiments also serve to verify the accuracy of the M-PWTD enhanced MOT solver. This is done by Fourier transforming far-field signatures that are obtained using the time domain solver, computing the frequency domain RCS from this data at a frequency, and finally, comparing these against the RCS data obtained using a frequency domain CFIE fast multiple code (FISC) [53]. In all the examples presented herein, the incident pulse is a modulated Gaussian plane wave that is parameterized as

$$E'(r,t) = \hat{p} \cos \left( 2\pi f_0 (t - r \cdot \hat{k}/c) \right) e^{-\frac{(t - r \cdot \hat{k}/c - t_p)^2}{2\sigma^2}}, \quad (5.39)$$

where \( f_0 \) is the pulse's center frequency, \( \hat{k} \) and \( \hat{p} \) denote its direction of travel and polarization, \( \sigma = 6/(2\pi f_{bw}) \), and \( t_p = 3.5\sigma \). The parameter \( f_{bw} \) will henceforth be referred to as the "bandwidth" of the signal. It is to be noted that the power in the incident pulse at \( f = f_0 \pm f_{bw} \) is down by 160 dB relative to that at \( f_0 \).

In the first experiment, scattering from a VY218 aircraft, discretized using 9747 spatial basis functions, is analyzed using the ML-PWTD algorithm, and results are compared to those obtained earlier using the two-level PWTD scheme as well as FISC. The incident field travels along \( \hat{k} = -\hat{y} \), is polarized along \( \hat{p} = \hat{x} \), has a center frequency \( f_0 = 100 \) MHz, and a bandwidth \( f_{bw} = 60 \) MHz. Figure 5.12 compares the RCS patterns in the xy-plane at 75, 85, 110, 130 MHz. From these figures, it is apparent that the results obtained using the multilevel algorithm agree very well with those obtained using the two-level scheme and with FISC.
Figure 5.12: RCS patterns of a VFY218 in the xy-plane at (a) 75 MHz, (b) 85 MHz, (c) 110 MHz, and (d) 130 MHz.
Figure 5.13: RCS patterns of VFY218 in the xy-plane at (a) 212 MHz, (b) 244 MHz, (c) 276 MHz, and (d) 292 MHz.

In the second example, scattering is analyzed from a VFY218 that is electrically larger than the one studied earlier. In this case, the VFY218 is discretized using 45492 spatial unknowns. The incident field travels along \( \hat{k} = -\hat{y} \), is polarized along \( \hat{p} = \hat{x} \), has a center frequency \( f_0 = 250 \) MHz, and a bandwidth \( f_{bw} = 110 \) MHz. Figure 5.13 compares the RCS patterns at 212, 244, 276, and 292 MHz. These plots show that the ML-PWTD results agree well with those obtained using FISC. Also, in Figure 5.14 four snapshots of the magnitude of the
current on the surface are plotted. The first of these is almost immediately after the pulse is incident on the aircraft. The next two plots indicate that resonances along the length and breadth of the plane are set up. Finally, the last figure shows that while the currents on the rest of the body die down to about $-80$ dB below the peak, the currents around the engine inlet area are only down by $-30$ dB below the peak at $t = 82$ ns. Of course, as is to be expected, the currents at all locations on the aircraft fall below $-50$ dB after $t = 114$ ns (not shown in figure).

Figure 5.14: The magnitude of the current on VFY218 at four different snapshots. Assuming that at $t = 3.8 \times 10^{-10}$ s, the wave has just reached the leading wing, the snapshots are at (a) $t = 3.8$ ns, (b) $t = 38$ ns, (c) $t = 64$ ns, (d) $t = 82$ ns, and the colorbar shows the dB scale.
Figure 5.15: RCS patterns of the cone-sphere in the $yz$-plane at (a) 2.14 GHz, (b) 2.65 GHz, (c) 3.17 GHz, and (d) 3.79 GHz.

Next, scattering from a cone-sphere is analyzed. The cone is 1 m long and the radius of the half sphere attached to the cone is 0.235 m. The incident pulse has a center frequency of $f_0 = 3$ GHz and a bandwidth of $f_{bw} = 2.5$ GHz. A modulated Gaussian plane wave is incident upon the scatterer from the $\hat{k} = -\hat{y}$ direction and polarized along $\hat{p} = \hat{z}$. This cone-sphere is discretized using 75585 spatial basis functions. The RCS pattern in the $yz$-plane is computed and compared to those obtained using FISC in Figure 5.15 at 2.15, 2.65, 3.17, and 3.79 GHz. As is evident from these figures, all four results agree well with the frequency domain RCS data.
Figure 5.16: RCS patterns of the almond in the xz-plane at (a) 210 MHz, (b) 260 MHz, (c) 310 MHz, and (d) 380 MHz.

In the last example, scattering is analyzed from a NASA almond whose current is discretized using 105794 spatial basis functions. The scatterer fits in a box of dimension \(10 \times 3.66 \times 1.288 \text{ m}\), and is excited by an electromagnetic pulse that is incident upon it in the \(\hat{\mathbf{k}} = -\hat{z}\) direction, and is \(\hat{\mathbf{p}} = \hat{x}\) polarized. The incident pulse has a center frequency \(f_0 = 300\) MHz and bandwidth \(f_{bw} = 200\) MHz. The RCS patterns are computed in the xz-plane and compared to those obtained using FISC. From Figure 5.16 it is apparent that both the frequency and time domain results agree very well with each other.

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Finally, the predicted computational complexity and the memory requirement of the ML-PWTD enhanced MOT scheme is experimentally verified. The results that are presented herein are obtained using an SGI Origin 2000 machine with a peak performance rating of 360 Mflops. In Figure 5.17(a), the logarithm of the average time required to compute the interactions at one time step is plotted against $\log N_s$. Similarly, the logarithm of the memory required for the analysis is plotted against $\log N_s$ in Figure 5.17(b). The slopes of the lines provided for comparison are 2.00 for the direct MOT method and 1.00 for the ML-PWTD. Both these figures indicate that the experimentally obtained computational complexity and the memory requirements for the analysis follow the predicted scaling law. Indeed, it is apparent from these figures that the breakeven point with the classical MOT solver is as low as 1000 spatial unknowns.

![Graphs](image)

Figure 5.17: Computational complexity and memory scaling for the ML-PWTD enhanced MOT solver.

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5.5 Conclusions

In this chapter, incorporation of the nonwindowed two-level and multilevel PWTD schemes into an MOT solver for analyzing electromagnetic wave scattering from PEC objects is elucidated. To this end, a plane wave expansion that can be implemented as a three-stage PWTD algorithm is introduced for the RTBI associated with the time domain CFIE of Section 2.4. In this expansion, it becomes clear that only two-component vectors were needed to represent the plane wave spectrum of electromagnetic fields. The implementation of the PWTD algorithm within the framework of the MOT solver was reviewed both for the two-level and multilevel schemes. Numerical results were presented demonstrating the accuracy and efficiency of the proposed schemes. Through these numerical results, the predicted $O(N_t N_s^{1.5} \log N_s)$ and $O(N_t N_s \log^2 N_s)$ computational complexities for the two-level and multilevel PWTD enhanced MOT schemes, respectively, were also verified. When compared to the classical MOT scheme, it was seen that the two-level scheme becomes more advantageous to employ for $N_s > 700$ whereas this breakeven point occurs at $N_s = 1000$ for the multilevel scheme. With these methods, analysis of transient scattering from realistic structures finally seems within the reach of current computational resources.
CHAPTER 6
EFFICIENT IMPLEMENTATION OF EXACT BOUNDARY CONDITIONS IN
FINITE-DIFFERENCE TIME-DOMAIN SIMULATIONS

6.1 Introduction

Previous chapters have dealt with the integral equation (IE)-based methods for analyzing transient wave scattering in an unbounded medium. These methods have certain advantages over those based on differential equations when the scatterer under study is impenetrable or homogeneous, i.e., when the scatterer-wave interaction can be completely characterized with a single set of boundary conditions on the scatterer surface. However, when the scatterer is inhomogeneous and confined in a region bounded by simple boundaries, differential equation (DE)-based methods prove to be more efficient and easier to apply. There is a third class of problems in which an inhomogeneous scatterer resides in a homogeneous unbounded medium. Although this class of problems is difficult to handle by either method alone, a hybrid method which uses a DE-based technique to analyze the wave-structure interaction and an IE-based technique to enforce the radiation boundary condition is capable of performing the analysis gracefully. Since no theoretical approximations are made in imposing the radiation condition in this manner, this method has been referred to as the exact boundary condition (EBC) in literature. With its exorbitant computational needs, evaluation of the retarded-time boundary integrals (RTBIs) in the IE-based part has been responsible for the scarce attention the EBC has received. In this chapter, armed with PWTD schemes that help reduce the computational costs of evaluating RTBIs, we turn our attention to such a hybrid method. Due to its ease of implementation, we will concentrate on a scheme based on a finite-difference time-domain (FDTD) algorithm, and demonstrate how the computational complexity of imposing the EBC
can be reduced using the PWTD enhanced schemes introduced earlier. However, the ideas presented here can be easily generalized for techniques based on other DE-based methods such as time-domain finite-element or finite volume methods.

6.2 Problem Definition and Solution Approach

The FDTD method is a powerful tool for transient analysis of linear wave phenomena involving complex inhomogeneous bodies. However, when the structure of interest resides in an unbounded medium, truncating the computational domain requires special attention. Mesh truncation schemes incorporated into FDTD schemes as nonreflecting or absorbing boundary conditions (NRBCs or ABCs) can be classified with respect to their locality in space and time. Local schemes such as those due to Mur [21], Higdon [22], and Bayliss and Turkel [20] are either based on approximations to the one-way wave equation or the Sommerfeld radiation condition. These NRBCs relate each field value sampled on the truncation boundary to the nearby fields that lie inside the computational domain and require a small and constant amount of operations to update the boundary fields. That is, updating each boundary field value requires $O(1)$ operations. Denoting the number of field samples that lie on the truncation boundary by $N_s$ and the number of time steps used in the analysis by $N_t$, the cost of imposing these local NRBCs scales as $O(N_tN_s)$. Another type of local mesh truncation scheme wraps absorbing materials around the computational domain. This kind of mesh truncation has gained popularity with the advent of perfectly matched layers (PML) that are (theoretically) reflectionless materials mathematically constructed for numerical simulations [23, 24]. Computational complexity of the PML technique also scales as $O(N_tN_s)$.

Two drawbacks impede the applicability of local NRBCs. First, a buffer region has to be introduced between the scatterer and the truncation boundary. The size of the buffer region can be considerably large in NRBCs based on the one-way wave equation. In fact, a study by Ting
and Miksis [15] compares the computational requirements of the EBC method to those of these local methods, and concludes that the EBC “method is more efficient in the sense that for the same degree of accuracy it requires much less storage space and less operations per time step.” Such an argument cannot be made with respect to the PML approach, because contrary to the other local NRBCs, no approximations are involved in its theoretical development. Nevertheless, a small buffer region still helps decrease the inaccuracies associated with the discrete implementation of PML [107]. A second drawback is that local NRBCs are not applicable on concave truncation boundaries and can inflate the size of the computational domain unnecessarily. This drawback becomes more pronounced in analyzing scattering from concave slender bodies (see Figure 6.1(a)) such as material coated semi-open cavities. In such a case, although the scatterer volume scales as $O(N_s)$, the computational domain must contain $O(N_s^{1.5})$ field samples. Hence, even though updating the fields on the truncation boundary requires $O(N_tN_s)$ operations, the $O(N_tN_s^{1.5})$ cost of updating the fields in the computational volume determines the computational complexity of the algorithm.

On the other hand, the nonlocal EBCs based on Huygens’ equivalence principle can be imposed on arbitrary truncation boundaries (Figure 6.1(b)) reducing the cost of updating the fields in the computational domain to $O(N_tN_s)$ [14-18]. However, since RTBIs have to be evaluated in order to update each field value on the truncation boundary, the cost of imposing this boundary condition scales as $O(N_tN_s^2)$. Clearly, this unacceptable cost can be reduced to $O(N_tN_s\log^2 N_s)$ by using the multilevel PWTD schemes described in the previous chapters.
In this chapter, the elements in implementing reduced complexity EBCs for acoustics and electromagnetics will be elucidated in Sections 6.3 and 6.4, respectively. Numerical results verifying the expected cost scalings will be presented in each section separately and conclusions for the whole chapter will be drawn in Section 6.5. However, two points should be mentioned before moving on to these sections. First, although there are no approximations involved in the theoretical development of the EBCs, discretization of fields and equivalent sources as well as accumulation of grid dispersion in the FDTD domain [108, 109] introduces inaccuracies in the numerical implementation. Second, in cases where the FDTD mesh has to be truncated on a convex boundary, the size of the computational domain scales as $O(N_s^{1.5})$ and the computational complexity can at best be $O(N_t N_s^{1.5})$. Although the technique introduced in this chapter would still reduce the cost of implementing EBCs to $O(N_t N_s \log^2 N_s)$, i.e., lower than that of updating
the internal fields, it still proves to be more expensive than applying local boundary conditions. Hence, the efficacy of the introduced results will only be demonstrated for concave slender scatterers.

6.3 Application to Acoustic Finite-Difference Time-Domain Method

The FDTD method for analyzing acoustic waves has been used in tackling a large variety of problems involving nonlinear, lossy, and dispersive structures [110-113]. In fact, improvements in the acoustical FDTD methods have gone hand-in-hand with those in the electromagnetics community [114, 115]. However, due to the popularity of finite element methods [13], FDTD—as well as the closely related finite volume technique [110]—has not enjoyed widespread use in the acoustics community. Nevertheless, mesh truncation schemes have always been of primary interest as most techniques developed for either of these differential equation-based techniques can, in theory, be applied within the framework of the other. This section deals with such a technique, namely global mesh truncation using an EBC, and shows how the computational burden associated with this otherwise excellent scheme can be alleviated using the PWTD algorithm. The FDTD scheme has been chosen for its ease of implementation. However, the ideas presented in this section can be easily applied to any differential equation-based method. This section is organized as follows. In Section 6.3.1, the FDTD method as formulated by Botteldooren [110] will be briefly reviewed and an EBC based on RTBIs will be introduced. The plane wave expansions that facilitate the application of the three-stage PWTD algorithm to the involved RTBIs will be derived in Section 6.3.2. How the resulting PWTD algorithm can be incorporated into the FDTD scheme will be elucidated and the complexity estimates of the resulting scheme will be derived in the same section. Finally, some numerical examples demonstrating the efficacy of the proposed method will be presented in Section 6.3.3.
6.3.1 FDTD formulation

The time-dependent acoustic pressure and velocity fields, denoted by \( p(r, t) \) and \( v(r, t) = \dot{x}v_x(r, t) + \dot{y}v_y(r, t) + \dot{z}v_z(r, t) \), in a lossless medium are related by the equations [80, 110, 113]

\[
\partial_t v(r, t) = -\frac{1}{\rho(r)} \nabla p(r, t), \tag{6.1}
\]

\[
\partial_t p(r, t) = -\frac{1}{\kappa(r)} \nabla \cdot v(r, t), \tag{6.2}
\]

where \( \rho(r) \) and \( \kappa(r) \) are the position-dependent mass density and compressibility of the medium, respectively, and \( \partial_t \) denotes partial differentiation with respect to \( t \). The speed of sound at a point in the medium is given by \( c(r) = [\rho(r)\kappa(r)]^{-1/2} \). In order to numerically solve Eqs. (6.1) and (6.2) within a volumetric domain \( V \) subject to initial and boundary conditions specific to the problem of interest, first \( V \) is subdivided into rectangular cells. In the resulting computational lattice, the pressure field is sampled at the corners of the cells and the components of the velocity field at the centers of respective cell edges as shown in Figure 6.2. The temporal sampling of fields is also staggered in time by sampling \( p(r, t) \) at time points \( t = n\Delta_t \) and \( v(r, t) \) at \( t = (n + 1/2)\Delta_t \) for a time step size of \( \Delta_t \) and integer \( n \). In what follows, let \( \Delta_{\xi}, \xi = x, y, z \), denote the dimension of the cells along the coordinate axes, the triplet \((i, j, k)\) denote the location \((i\Delta_x, j\Delta_y, k\Delta_z)\), and a superscript to a field quantity denote the time step at which it is sampled. To further simplify the notation, let an arrow over an integer quantity indicate an increment by \( 1/2 \), i.e., \( \vec{n} = n + 1/2 \). Using this notation and approximating the partial derivatives in Eqs. (6.1) and (6.2) by central difference formulas yields the update equations [110]

\[
v^n_x(i, j, k) = v^{\vec{n}-1}_x(i, j, k) - \frac{\Delta_t}{\Delta_x \rho(i, j, k)} \left[ p^n(i + 1, j, k) - p^n(i, j, k) \right], \tag{6.3}
\]
\begin{align}
\nu_y^n(i, j, k) &= \nu_y^{n-1}(i, j, k) - \frac{\Delta t}{\Delta y \rho(i, j, k)} \left[ p^n(i, j + 1, k) - p^n(i, j, k) \right], \tag{6.4} \\
\nu_z^n(i, j, k) &= \nu_z^{n-1}(i, j, k) - \frac{\Delta t}{\Delta z \rho(i, j, k)} \left[ p^n(i, j, k + 1) - p^n(i, j, k) \right], \tag{6.5} \\
p^{n+1}(i, j, k) &= p^n(i, j, k) - \frac{\Delta t}{\kappa(i, j, k)} \left[ \frac{\nu_x^n(i, j, k) - \nu_x^n(i, j, k)}{\Delta x} + \frac{\nu_y^n(i, j, k) - \nu_y^n(i, j, k)}{\Delta y} \\
&\quad + \frac{\nu_z^n(i, j, k) - \nu_z^n(i, j, k)}{\Delta z} \right]. \tag{6.6} 
\end{align}

Note that Eqs. (6.3)-(6.6) indicate that each velocity field component is updated using two nearest pressure field values and each pressure field value is updated using six nearest velocity field components.

![Figure 6.2: Locations of the field sampling points over a unit FDTD cell.](image)

The boundaries of the computational domain on which the EBC is going to be imposed can be chosen in many different ways. In this study, the computational domain is truncated such that the boundaries either coincide with the faces of the FDTD cells as shown in Figure 6.3(a) or they cut halfway through the FDTD cells as in Figure 6.3(b). In the former case, some pressure sampling points lie on the truncation boundary $S_p$ and one or more of the velocity field

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Figure 6.3: Two types of truncation boundary: (a) boundary $S_p$ coincides with the faces of the FDTD cells, and (b) boundary $S_v$ cuts halfway through the FDTD cells.

components needed for updating the values of these pressure points are not available. This kind of grid truncation is ideal if $S_p$ coincides with a pressure release surface on which $p(r, t) = 0$. In the latter case, only the components of the velocity field normal to the truncation boundary $S_v$ lie on it and cannot be updated using the regular equations. This kind of grid truncation is ideal for terminating the FDTD mesh at a rigid surface on which the normal component of $v(r, t)$ is zero. However, in either case, if the domain of analysis is assumed to reside in an unbounded homogeneous space with parameters $\rho_0$ and $\kappa_0$, then the field values of the boundary have to be found using an NRBC that emulates the radiation condition. As its name implies, this kind of boundary condition lets the energy impinging on it from within the computational domain exit the domain with no reflections. One way to impose such a boundary condition is to evaluate the values of the fields on the truncation boundary exactly from the field values that lie strictly in $V$ by using the Kirchhoff-Helmholtz integral theorem [80]. This theorem, which is intimately related to Huygens' equivalence principle [116], helps express the field values exterior to some closed surface $S_s$ in terms of the field values on this surface when the exterior of $S_s$ is an unbounded homogeneous space. More specifically, choosing $S_s$ to lie in $V$, the field values on the truncation boundaries are given as
\[ p(r,t) = \int_0^t dt' \left[ \mathcal{L}_{pp}\{p(r,t')\} + Z_0 \mathcal{L}_{pv}\{v(r,t')\} \right] \quad r \in S_p, \tag{6.7} \]

\[ \hat{n} \cdot v(r,t) = -\int_0^t dt' \left[ Z_0^{-1} \mathcal{L}_{vp}\{p(r,t')\} + \mathcal{L}_{vv}\{v(r,t')\} \right] \quad r \in S_v, \tag{6.8} \]

where

\[ \mathcal{L}_{pp}\{p(r,t)\} = \int_{S_t} dr' \hat{n}' \cdot \nabla G(r, r', t) \cdot p(r', t), \]

\[ \mathcal{L}_{pv}\{v(r,t)\} = \int_{S_t} dr' c_0^{-1} \nabla^2 G(r, r', t) \cdot \hat{n}' \cdot v(r', t), \]

\[ \mathcal{L}_{vp}\{p(r,t)\} = \int_{S_t} dr' c_0 \left[ \hat{n} \cdot \nabla \hat{n}' \cdot \nabla' G(r, r', t) \right] \cdot p(r', t), \]

\[ \mathcal{L}_{vv}\{v(r,t)\} = \int_{S_t} dr' \hat{n} \cdot \nabla G(r, r', t) \cdot \hat{n}' \cdot v(r', t). \tag{6.9} \]

In the above equations \( \hat{n} \) and \( \hat{n}' \) denote outward unit normals to surfaces \( S_v \) and \( S_s \), respectively, \( G(r, r', t) = (4\pi |r - r'|)^{-1} \delta(t - c_0^{-1} |r - r'|) \) is the free space Green’s function for a medium with wave speed \( c_0 = (\rho_0 \kappa_0)^{1/2} \) and intrinsic impedance \( Z_0 = c_0 \rho_0 \), and \( \ast \) denotes temporal convolution. In addition, the lower limit on temporal integrations is set to zero assuming that \( p(r,t) = 0 \) and \( v(r,t) = 0 \) for \( t \leq 0 \) in \( V \). Note that Eq. (6.7) can be simplified such that it involves no temporal integration. However, the present form provides a degree of symmetry with Eq. (6.8) that will prove useful in the upcoming discussions.

Several remarks about the properties and proper usage of Eqs. (6.7)-(6.9) are in order. The linear (integro-differential) operators \( \mathcal{L}_{\xi\zeta}\{\} \), \( \xi, \zeta \in \{p,v\} \), are generally referred to as retarded-time boundary integrals (RTBIs) due to the delay term \( \delta(t - c_0^{-1} |r - r'|) \) that appears in their kernels. If the closest distance between the integration surface \( S_s \) and truncation surfaces \( S_p \) and \( S_v \) is chosen to be greater than \( c_0 \Delta t \), then due to the time-retardation the field values on \( S_p \)
and $S_v$ at the next time step can be calculated from the current and past field values on $S_s$. Keeping the integration and truncation boundaries separated also guarantees that the integrands of the RTBIs never become singular and obviates the need for using principle value or finite part integrations. However, as will be elucidated shortly, evaluation of these RTBIs is a computationally expensive process. This is a consequence of the fact that in the evaluation of each truncation boundary field value, all the field values on the integration boundary contribute. Hence, the global (or nonlocal) nature of this type of boundary condition. Finally, it is well-known that the integral operators on the right-hand sides of Eqs. (6.7) or (6.8) have a null-space, and therefore using either one of these equations alone can yield results that are corrupted by spurious oscillations excited by numerical inaccuracies [76]. This phenomenon is closely related to the internal resonance problem addressed in Chapter 2, and it can also be prevented by using a linear combination of both equations. To this end, assuming that $S_p$ is placed half a cell outside of $S_v$, first, the field values on both these truncation surfaces are calculated using Eq. (6.7) and (6.8). Then, the velocity field values on $S_v$ are calculated again using the usual FDTD update, which makes use of the pressure field values on $S_p$, and the final field values on $S_v$ are set to the arithmetic average of the two results.

In the present study, the NRBC based on Eqs. (6.7)-(6.8), henceforth referred to as EBC, is incorporated into the FDTD scheme as follows. The integration surface $S_s$ is chosen such that it coincides with the faces of the FDTD cells. Hence, the samples of $p(r, t)$ already lie on $S_s$ at points, say $r_m \in S_s$ for $m = 1, \ldots, N_s$, and the values of $\mathbf{n} \cdot \mathbf{v}(r, t)$ on $S_s$ are interpolated from the nearest available samples using third-order Lagrange interpolation in three dimensions to $r_m$. In order to ensure that the closest distance between $S_s$ and the truncation surfaces is greater than $c_0 \Delta t$, $S_v$ and $S_p$ are placed 1½ and 2 cells away from $S_s$, respectively. In order to discretize the RTBIs, a bilinear interpolation function with rectangular support is associated with each of
the field samples on $S_s$. This is equivalent (in boundary element terminology) to expanding the fields on the integration surface in terms of $N_s$ spatial basis functions $f_m(r)$ as

$$p(r, t) = \sum_{m=1}^{N_s} p(r_m, t) f_m(r),$$  \hspace{1cm} (6.10)$$

$$\hat{n} \cdot v(r, t) = \sum_{m=1}^{N_s} v_{\perp}(r_m, t) f_m(r),$$  \hspace{1cm} (6.11)$$

where $p(r_m, t)$ and $v_{\perp}(r_m, t)$ are the time-dependent pressure and normal velocity field component expansion coefficients associated with $f_m(r)$. Henceforth, $\hat{v}$ will be used to denote the component of the velocity field normal to the associated surface $S_s$, $S_p$, or $S_v$. The basis function $f_m(r)$ is the bilinear surface element [76] centered at $r_m$ with side length $2\Delta_\xi$ along the $\xi \in \{x, y, z\}$ direction. Substituting Eqs. (6.10) and (6.11) in Eqs. (6.7) and (6.8) and evaluating the observed fields at locations $r_m'$ on $S_p$ or $S_v$ yields

$$p^{n'+1}(r_m') = \int_0^{(n'+1)\Delta_t} dt' \sum_{m=1}^{N_s} \left[ \mathcal{L}_{pp}\{f_m(r)p(r_m, t)\} + Z_0 \mathcal{L}_{pv}\{f_m(r)\hat{v}(r_m, t)\} \right]_{r=r_m'},$$  \hspace{1cm} (6.12)$$

$$\hat{v}^{n'+1}(r_m') = -\frac{\tilde{n} \Delta_t}{\alpha} \sum_{m=1}^{N_s} \left[ Z_0^{-1} \mathcal{L}_{vp}\{f_m(r)p(r_m, t)\} + \mathcal{L}_{vv}\{f_m(r)\hat{v}(r_m, t)\} \right]_{r=r_m'},$$  \hspace{1cm} (6.13)$$

where it is implicitly assumed that each $r_m'$ corresponds to a triplet $(i, j, k)$. In Eqs. (6.12) and (6.13), the temporal variations of the fields on $S_s$ are approximated by one-sided cubic interpolants as given in Eq. (2.15), and the temporal integrations are evaluated using a cubic approximation to the integrand. Note that evaluating the field value at a single location on the truncation boundary using Eqs. (6.12) and (6.13) requires $O(N_s)$ operations. Since there are $O(N_s)$ field points on the truncation boundary, for which these equations have to be invoked at each time step, the computational complexity of updating the boundary fields for $N_t$ time steps
scales as $O(N_t N_s^2)$. Clearly, the origin of this high complexity is the same as that responsible for the high complexity of the classical MOT schemes. Hence, it can be reduced to as low as $O(N_t N_s \log^2 N_s)$ using the multilevel PWTD enhanced schemes introduced in the previous chapters. Before discussing how the PWTD algorithm can be adapted for the present case, a note about the density of field samples is in order. In order to achieve reasonable accuracy with an FDTD simulation, the edges of each cell should be as small as $\lambda/20$, where $\lambda$ denotes the shortest wavelength in the computational domain, whereas a sampling density of $\lambda/10$ yields accurate results for integral equation-based methods when piecewise linear spatial basis functions are used [117]. Hence the cost of imposing the NRBC presented above can be reduced by using larger basis functions on the integration boundary and by evaluating the fields at a fewer number of points on the truncation boundaries and interpolating to FDTD sample points. This reduction in cost, however, does not reduce the computational complexity of the NRBC, and can be used both for the classical method described above and the accelerated implementation discussed next.

6.3.2 Efficient evaluation of boundary fields using the PWTD algorithm

The PWTD algorithm helps reduce the computational complexity of evaluating fields that can be expressed in the form $\mathcal{L}\{h(r,t)\}$, where $\mathcal{L}$ is an RTBI operator acting on the source distribution $h(r,t)$. In Section 3.3.1, the three-stage PWTD algorithm was elucidated for the RTBI whose kernel was the free space Green's function. How this PWTD algorithm could be utilized in two-level and multilevel settings to speed up the computation of $\mathcal{L}\{h(r,t)\}$ was also discussed in Chapter 3. In the following chapters, it was shown that the two-level and multilevel schemes based on the PWTD algorithm could also be employed for RTBIs whose kernels could be obtained from the free space Green's function by temporal and spatial differentiations. The key to application of this acceleration technique lies in being able to express $\mathcal{L}\{h(r,t)\}$ in terms
of a plane wave expansion such as in Eqs. (4.19)-(4.21). Before deriving the plane wave expansions for the RTBIs in Eqs. (6.9), let us first recall the setting in which expansions akin to Eqs. (4.19)-(4.21) hold. Referring to Figure 4.1(a), assume that a basis function \( f_m(r) \) with a time signature \( q(t) \) associated with it is enclosed inside a sphere of radius \( R_s \) and that the field \( \mathcal{L}\{ y(r, t) \} \) due to the source distribution \( y(r, t) = f_m(r)q(t) \) is observed at a point \( r_m' \) inside a second sphere of equal radius. Let \( R_c \) denote the magnitude of the vector \( R_c \) pointing from the center of the source sphere to the center of the observation sphere. Further assume that the two spheres do not overlap, i.e., \( R_c > 2R_s \) and that the source signature \( q(t) \) is partitioned into \( L \) subsignals \( q_l(t), \ l = 1, \ldots, L, \) such that each subsignal \( q_l(t) \) starts at \( t = t_l^{min} \) and ends at \( t = t_l^{max} \) with \( t_l^{max} - t_l^{min} \leq T_s = (R_c - 2R_s)/c_0. \) That is,

\[
q(t) = \sum_{l=1}^{L} q_l(t) \quad ; \quad y(r, t) = \sum_{l=1}^{L} y_l(r, t). \tag{6.14}
\]

Such a partitioning of \( q(t) \) into subsignals, which satisfy the finite duration constraint and which are approximately bandlimited to \( \omega_s = \chi_1 \omega_{max} \) for \( \chi_1 > 1, \) was described in Section 3.5.2. Under these spatial and temporal restrictions, the plane wave expansions for \( \mathcal{L}_{\xi\zeta}\{ y_l(r, t) \}, \ \xi, \zeta \in \{ p, n \}, \) can be obtained by substituting Eq. (4.16) in place of the free space Green's functions in Eqs. (6.9), and following the discussions in 3.3.2 to restrict the number of plane waves needed. Using the identity \( f(r_m') = \int_S dr \ f(r) \tilde{f}_m'(r) , \) where \( \tilde{f}_m'(r) = \delta(r - r_m') \) and \( f(r) \) is an arbitrary function of space and keeping in mind that \( q(t) \) denotes \( p(r_m, t) \) if \( \zeta = p \) and \( \tilde{v}(r_m, t) \) if \( \zeta = n, \) the resulting identities can be succinctly expressed as

\[
\mathcal{L}_{\xi\zeta}\{ y_l(r, t) \}\big|_{r=r_m'} = \begin{cases} 0 & ; \ t < t_l^{max} \\ \tilde{\mathcal{L}}_{\xi\zeta}\{ y_l(r, t) \}\big|_{r=r_m'} & ; \ t \geq t_l^{max} \end{cases}. \tag{6.15}
\]

where
\[
\tilde{L}_\xi \zeta (\gamma_t(r,t)) \big|_{r=r_m} = \sum_{p=0}^{K} \sum_{q=-K}^{K} w_{pq} \mathcal{P}_\xi^- (\hat{k}_{pq}, \tilde{f}_{m',t}) \ast \mathcal{F}(\hat{k}_{pq}, R_c, t) \ast \mathcal{P}_\zeta^+ (\hat{k}_{pq}, f_m, t) \ast q_l(t).
\]

In Eq. (6.16), \( K = \left[ 2 \chi_1 \chi_2 R_s \omega_{\text{max}} / c_0 \right] \) (\( \chi_2 > 1 \) is the excess bandwidth factor \([99]\)), the weights \( w_{pq} \) and the directions \( \hat{k}_{pq} \) are as given in Eq. (3.21), the translation function \( \mathcal{F}(\hat{k}, R_c, t) \) is given by

\[
\mathcal{F}(\hat{k}, R_c, t) = \begin{cases} 
0 & ; \ |t| > R_c / c \\
\frac{\partial^2}{16 \pi^2 c R_c} \sum_{k=0}^{K} (2k+1) P_k \left( \frac{ct}{R_c} \right) P_k \left( \hat{k} \cdot \frac{R_c}{R_c} \right) & ; \ |t| \leq R_c / c
\end{cases},
\]

with \( P_k(\cdot) \) denoting the Legendre polynomial of degree \( k \), and the projection operators are defined as

\[
\begin{align*}
\mathcal{P}_p^+ (\hat{k}, f_m, t) &= \int_{S_\Sigma} \text{d}r' (\hat{k} \cdot \hat{n}) f_m(r') \delta \left[ t + \hat{k} \cdot (r' - r_c) / c_0 \right] \\
\mathcal{P}_v^+ (\hat{k}, f_m, t) &= \int_{S_\Sigma} \text{d}r' f_m(r') \delta \left[ t + \hat{k} \cdot (r' - r_c) / c_0 \right] \\
\mathcal{P}_p^- (\hat{k}, \tilde{f}_{m', t}) &= \int_{S_p} \text{d}r \tilde{f}_{m'}(r) \delta \left[ t - \hat{k} \cdot (r - r_c) / c_0 \right] \\
\mathcal{P}_v^- (\hat{k}, \tilde{f}_{m', t}) &= \int_{S_v} \text{d}r (\hat{k} \cdot \hat{n}) \tilde{f}_{m'}(r) \delta \left[ t - \hat{k} \cdot (r - r_c) / c_0 \right],
\end{align*}
\]

with \( r_c \) denoting the center of the sphere in which \( f_m(r') \) or \( \tilde{f}_{m'}(r) \) resides.

Clearly, Eqs. (6.15) and (6.16) permit the evaluation of the fields \( L_\xi \zeta \{ \cdot \} \), \( \xi, \zeta \in \{ p, v \} \), using a three-stage PWTD algorithm consisting of (i) calculation of outgoing rays, (ii) translation of these rays into incoming rays at \( t = t_{\text{max}} \) (implicitly implementing the time gating in Eq. (6.15)), and (iii) projection of the incoming rays onto observer locations. However, using this PWTD algorithm in efficient evaluation of the fields in Eqs. (6.12) and (6.13) requires a setup.
stage. In this preprocessing stage, both space and time must be partitioned into a multilevel structure in order to accommodate the space-time restrictions associated with the plane wave expansions introduced above. A multilevel partitioning scheme to serve this purpose was described in detail in Section 4.4.1. In this scheme, spatial partitioning is accomplished by first enclosing $S_s$, $S_p$, and $S_v$ inside a cubical box and then recursively subdividing this box into smaller and smaller boxes. The number of recursions $N_l$ determines the number of levels in the multilevel structure, in which a box at a given level consists of eight cubical boxes that belong to the finer level. The set of basis functions or the set of observation locations that fall within the same box is termed a group. Note that each of the groups at a level $v$ can be enclosed in a sphere of radius $R_s(v)$ and that this radius doubles from one level to the next. The temporal partitioning is accomplished by first defining a fundamental duration $T_s(1)$ at level 1 (finest level) that is proportional to $R_s(1)/c_0$, and then obtaining the fundamental durations for the higher levels by doubling $T_s(\cdot)$ from one level to the next. With this space-time partitioning, the interactions between two groups $(\gamma_v, \gamma'_v)$ at level $v$, i.e., the field at each observation point in group $\gamma_v$ due to sources in group $\gamma'_v$ and vice versa, are evaluated as follows. If the distance between the centers of these groups, $R_{c,\gamma,\gamma'_v}$, is larger than $c_0T_s(v)+2R_s(v)$, then the three-stage PWTD algorithm associated with Eqs. (6.15)-(6.18) is used in evaluating the interactions between them. Otherwise, the interactions between these groups are accounted for by the pairwise interactions of their children in the next lower level. Eventually, the only interactions that are evaluated by direct application of Eqs. (6.12) or (6.13) are those that occur between two level 1 groups $(\gamma_1, \gamma'_1)$ with $R_{c,\gamma,\gamma'_1} < c_0T_s(1)+2R_s(1)$. This type of direct interaction is called a near-field interaction and all the other interactions evaluated via the PWTD algorithm are called far-field interactions. Note that, in this manner, the field associated with a basis function $f_m(r)$ and observed at $r_{m'}$ is evaluated either through a near-field interaction or through a far-field interaction at a single level by the formulas
\begin{align}
\rho^{n+1}(r_{m'}) &= \int_0^{(n'+1)\Delta_t} dt' \sum_{p=0}^{K} \sum_{q=-K}^{K} w_{pq} \mathcal{P}_p^{-}(\hat{k}_{pq}, \tilde{f}_{m'}, t') \ast T(\hat{k}_{pq}, t') \ast \\
&\quad \sum_{l} \left[ \mathcal{P}_p^{+}(\hat{k}_{pq}, f_{m}, t') \ast p_l(r_{m}, t) + Z_0 \mathcal{P}_v^{+}(\hat{k}_{pq}, f_{m}, t') \ast \hat{v}_l(r_{m}, t) \right], 
\end{align}

\begin{align}
\tilde{\rho}^{n'}(r_{m'}) &= \int_0^{\tilde{n}'\Delta_t} dt' \sum_{p=0}^{K} \sum_{q=-K}^{K} w_{pq} Z_0^{-1} \mathcal{P}_v^{-}(\hat{k}_{pq}, \tilde{f}_{m'}, t') \ast T(\hat{k}_{pq}, t') \ast \\
&\quad \sum_{l} \left[ \mathcal{P}_p^{+}(\hat{k}_{pq}, f_{m}, t') \ast p_l(r_{m}, t) + Z_0 \mathcal{P}_v^{+}(\hat{k}_{pq}, f_{m}, t') \ast \hat{v}_l(r_{m}, t) \right], 
\end{align}

where $K$, $w_{pq}$, and $\hat{k}_{pq}$ all depend on $R_s(v)$ ($v$ is the level at which the interaction occurs); the duration of the subsignals $p_l(r_{m}, t)$ and $\hat{v}_l(r_{m}, t)$ are proportional to $T_s(v)$; and the summation index $l$ should span only those signal intervals that produce a response at time $t'$. 

Three important points should be mentioned regarding Eqs. (6.19) and (6.20). First, the outgoing rays (the square bracketed terms)—and, hence, the translation functions—in both equations are the same. This is expected as outgoing rays define the time dependent radiation pattern of the sources that form the source group and, therefore, their form does not depend on the observation point. Second, the incoming rays, i.e., the result of convolution of the translation function with the outgoing rays, would be the same for all observation points that fall within the same group as $r_{m'}$ and can also be reused for calculating the fields throughout this group. Finally, as discussed in Section 3.4.2, at each level, the outgoing rays need not be evaluated directly from the source signatures and the incoming rays need not be projected directly onto the observer locations. A more efficient scheme would recycle information by interpolating and splicing the outgoing rays at a level to form the rays of the higher level, and by resecting and anterpolating the incoming rays at a level to form the rays of the lower level.
The ideas put forth thus far are systematically realized by the following algorithm that efficiently evaluates the fields on the truncation boundaries $S_p$ and $S_v$. Due to its similarity to the multilevel algorithm presented in detail in Section 4.4.2, the operations at each step will be briefly described and the computational complexity of each step will be stated without derivation.

1. At each time step, all the field values on $S_s$ are computed and stored, and all near-field contributions to $p^{n+1}(r_{m'})$ and $\tilde{v}^{n}(r_{m'})$ for all $r_{m'} \in \{S_p, S_v\}$ are evaluated via Eqs. (6.12) and (6.13), respectively. Due to the local nature of the operations in this step, it can be completed in $O(N_s)$ operations at each time step.

2. When the current time step is an integer multiple of $T_s(n)/\Delta_t$ for some level $n$, the outgoing rays for all groups at that level are computed. For the lowest level, this is done directly by evaluating the square bracketed term that appears in Eqs. (6.19) and (6.20). For higher levels, the outgoing rays are formed from the rays of one lower level through interpolation and resection operations. The most costly operation in this step is the interpolation operation, which can be performed in $O(N_s \log N_s)$ operations per level per time step using the fast spherical filter of Jacob-Chien and Alpert [102]. As there exist $O(\log N_s)$ levels the average cost of this step is $O(N_s \log^2 N_s)$ per time step.

3. Once the outgoing rays are formed at a level, translation of these rays, i.e., convolution of these rays with the translation functions, can be performed between groups that interact at that level. Translation operation can be performed most efficiently by first evaluating the Fourier transform of each outgoing ray using fast Fourier transforms (FFTs), then multiplying the resulting outgoing ray spectrum by the spectrum of the translation function, and finally inverse Fourier transforming the result. The resulting rays are superimposed on the incoming rays of the observation groups. Performing the FFTs proves to be the most costly operation.
in each ray translation, yielding an overall computational complexity of \(O(N_s \log^2 N_s)\) per time step for this step.

4. When the current time step is an integer multiple of \(T_s(v)/\Delta t\) for some level \(v\), the incoming rays at that level are obtained by resecting and interpolating the incoming rays at a higher level. Finally, the far-field contributions to \(p^{n'+1}(r_{m'})\) and \(\hat{\nu}^{n'}(r_{m'})\) are evaluated at every time step by delaying and weighing the incoming rays at level 1 as dictated by the projection operators \(P_{\zeta}^{-}(\tilde{k}, \tilde{f}_{m'}, t)\). This step, which is conceptually the transpose of forming the outgoing rays, can also be accomplished in \(O(N_s \log^2 N_s)\) operations.

Summing up the cost of each step, it is seen that the computational cost of this algorithm, which evaluates \(p^{n'+1}(r_{m'})\) and \(\hat{\nu}^{n'}(r_{m'})\) for all \(r_{m'} \in \{S_p, S_v\}\) for \(N_t\) time steps, scales as \(O(N_t N_s^2 \log^2 N_s)\). This computational complexity is significantly lower than the \(O(N_t N_s^2)\) complexity of the classical algorithms. As shown in Section 4.4.2, the memory requirements of this algorithm scale as \(O(N_t N_s)\) as compared to \(O(N_t N_s^2)\) memory scaling of classical algorithms that avoid calculating the (direct) interaction coefficients at each time step. As will be demonstrated next, this efficient utilization of computational resources, together with the fact that the truncation boundaries in this scheme can be chosen arbitrarily, makes the proposed EBC the method of choice for analyzing wave phenomena in the presence of thin concave structures.

6.3.3 Numerical results

In this section, accuracy of the proposed EBC is inspected and its predicted computational complexity verified. The results obtained via the global NRBC of this section are compared to those obtained using PML [113]. The parameters for the PMLs will be indicated with the notation PML(a,b,c), where the entries a, b, and c denote the number of layers, the maximum value of the "attenuation coefficient," and its variation (2 for quadratic, 3 for cubic, etc.).
respectively. In all the examples that follow, the parameters of the ambient medium are assumed to be \( c_0 = 343 \) m/s and \( \rho_0 = 1.2 \) kg/m\(^3\), and the FDTD cells are chosen to be cubic with edge length of \( \Delta = \Delta_x = \Delta_y = \Delta_z = 1.25 \) mm.

In the first numerical experiment, a transparent point source is placed at the center of a cubical computational domain that measures 20 cells on each side. The temporal signature of the source is chosen such that it radiates a pressure field \( p(\mathbf{r}, t) = \partial_t f(t - R/c)/(4\pi R) \), where \( R \) is the distance from the source to \( \mathbf{r} \) and

\[
f(t) = e^{-(t-t_0)^2/2\sigma^2}
\]  

(6.21)

is a Gaussian pulse with \( \sigma = 69 \times 10^{-6} \) s. The time step size is chosen to be \( \Delta_t = 2 \times 10^{-6} \) s. This computational domain is truncated by the EBC and the total energy in it is calculated every time step. The time evolution of energy, normalized to its peak value, is shown in Figure 6.4 with the label EBC1. Had the mesh truncation been perfect, the energy inside the domain would become negligible after a certain time. However, it is seen that some energy is left inside the computational domain although at a very low level. How much accuracy would be compromised by expanding the fields on the integration boundary using basis functions with side length \( 4\Delta \) and by evaluating the field at only every other point on the truncation surfaces and finding the rest of the field values by linear interpolation is tested next. The result of this test, which used considerably lower computational resources, is also plotted in Figure 6.4 as EBC2. As expected, slightly more energy remains in the computational domain. In order to assess the effectiveness of using the EBC, the same test was repeated with three different PMLs wrapped around the computational domain. The parameters of these PMLs were adjusted to yield minimum energy after time step 400 with a given number of layers. It is seen that the EBC1 performs as well as an 8-layer PML and EBC2 as well as a 6-layer PML. The 10-layer PML and the EBC with fewer samples (EBC2) will be used for comparison purposes in the examples to follow.
Figure 6.4: Normalized energy in a $20 \times 20 \times 20$ cell computational domain enclosing a point source and terminated with EBC and PMLs.

In the second example, the claim that the EBC can be imposed on arbitrary truncation boundaries is tested by analyzing transient scattering from an L-shaped rigid plate depicted in Figure 6.5. The arms of the plate are 4 cells wide and 12 cells long each. This structure is insonified by a Gaussian plane wave propagating in the \( \hat{k}^{inc} = (-\hat{y} - \hat{z})/\sqrt{2} \) direction whose pressure field is given by \( p(r,t) = f(t - \hat{k}^{inc} \cdot r/c) \). Here, the function \( f(\cdot) \) is given in Eq. (6.21) with \( \sigma = 69 \times 10^{-6} \) s, and the time step size was as before. First, the scatterer is enclosed in a $16 \times 22 \times 22$ cell domain which was wrapped in PML(10,5,1,4) and the pressure field values throughout the domain were recorded at each time step. The incident field for this case was imposed using a total field/scattered field formulation [11]. Then, the EBC2 was imposed on an L-shaped boundary as shown in Figure 6.5 and the calculations repeated. This time, the incident field was imposed by adding it to the field values on the truncation boundaries. Snapshots of the scattered pressure field calculated by both methods are shown in Figure 6.6. Clearly, the field distributions look identical even around the intruding part of the truncation surface for the EBC.
Figure 6.5: Geometry of the L-shaped rigid plate and enclosing $S_p$.

Figure 6.6: Scattered pressure field distribution around the L-shaped rigid plate at $t = 3 \times 10^{-4}$ s calculated using (a) PML(10,5,1,4) and (b) EBC2.

In the previous two cases, due to the small sizes of the computational domains, not many field values were evaluated via the PWTD algorithm. A third test was devised to quantify the difference between using a 10-layer PML and imposing the EBC on a concave boundary on which most of the fields are calculated via the PWTD algorithm. To this end, again scattering from an L-shaped penetrable structure was studied. With reference to Figure 6.7, the dimensions and the properties of the scatterer were chosen as $a = 16\Delta$, $b = 4\Delta$, $\rho = 4.8$, and $\kappa = \kappa_0$. The incident pulse was chosen as in the previous example except $\sigma$ was set to $107.3 \times 10^{-6}$ s to
insure a sufficiently dense sampling within the scatterer and $\Delta_t = 1.822 \times 10^{-6}$ s to ensure stability and sufficient temporal resolution. For the PML case, the scatterer was enclosed in a $26 \times 46 \times 46$ cell domain, and for the EBC calculations a $26 \times 32 \times 32$ cell part of this domain was removed so that $S_s$ and $S_p$ lay three and five cells away from the structure in all directions, respectively. The total and scattered pressure field values at $(x, y, z) = (8\Delta, 30\Delta, 2\Delta)$ are plotted in Figure 6.8. The $L_2$ norm of the difference in the total field normalized to the $L_2$ norm of the total field calculated using PML is $2.3 \times 10^{-3}$.

![Figure 6.7: Geometry of the L-shaped penetrable scatterer.](image-url)
Finally, the computational complexity of the PWTD-enhanced EBC method is verified. This is accomplished by analyzing plane wave scattering from the L-shaped penetrable scatterers defined with $\alpha$ values of $4\Delta$, $8\Delta$, $16\Delta$, $32\Delta$, and $64\Delta$. In each case $b = 4\Delta$ and $S_s$ is positioned 3 cells away from the scatterer in all directions. The CPU times and the memory used by imposing EBC without the PWTD, PWTD-enhanced EBC, and 10-layered PML for an analysis of 500 time steps are plotted with respect to the surface area of the outer truncation boundary $S_p$ in Figure 6.9, and are labeled as DIR, PWTD, and PML, respectively. In addition, reference lines with slopes 1.0, 1.5, and 2.0 are also drawn in these log-log graphs. Clearly, the PWTD algorithm has reduced the cost of imposing the EBC to scale almost linearly with $N_s$ per time step from $O(N_s^2)$. On the other hand, the CPU and memory requirements for the PML implementation seem to converge to the $O(N_s^{1.5})$ curve as expected. From Figure 6.9, it is predicted that the current implementation of PWTD-enhanced EBC will become more economical to use when $\alpha > 1100\Delta$. 

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Figure 6.9: Scaling of CPU time and memory for the EBC, PWTD-enhanced EBC, and PML methods.

6.4 Application to Electromagnetic Finite-Difference Time-Domain Method

This section outlines how the multilevel PWTD scheme can be used to reduce the computational complexity of imposing EBCs in FDTD simulation of electromagnetic phenomena. To this end, the standard FDTD method as introduced by Yee [118] is briefly reviewed, and how an EBC can be incorporated into this scheme is elucidated in Section 6.4.1. The plane wave expansions that facilitate the application of the three-stage PWTD algorithm to the involved RTBs will be derived and its incorporation into the evaluation of EBCs in a multilevel setting will be elucidated in Section 6.4.2. Finally, some numerical examples verifying the complexity of the proposed method will be presented in Section 6.4.3.

6.4.1 FDTD formulation

The implementation of the FDTD scheme does not depart from the norm [10, 11, 118]. The domain of interest is discretized into Yee cells with approximate side length of $\lambda/20$, where $\lambda$ is the wavelength associated with the highest frequency content of excitation $\omega_{max}$. As
illustrated in Figure 6.10, the electric and magnetic field components are sampled at the centers of the edges and faces of each cell, respectively, and are computed using the usual Yee scheme. This scheme can be arrived at by approximating the spatial and temporal derivatives in Maxwell's curl equations by centered finite differences. Denoting the time step size by $\Delta_t$ and the spatial sampling intervals by $\Delta_x$, $\xi = x, y, z$, let a triplet $(i, j, k)$ denote the location $(i\Delta_x, j\Delta_y, k\Delta_z)$ and a superscript to a field quantity the time step at which it is sampled. To further simplify the notation, let an arrow over an index indicate an increment by $1/2$. That is, $H_x^{\bar{n}}(i, j, k)$ means $x$ component of the $H$ field sampled at time $t = (n + \frac{1}{2})\Delta_t$ and location $r = (i\Delta_x, (j + \frac{1}{2})\Delta_y, (k + \frac{1}{2})\Delta_z)$. Using this notation, FDTD update equations for the $x$ components of the electric and magnetic fields can be written as follows:

$$
E_x^{n+1}(i, j, k) = E_x^n(i, j, k) + \frac{\Delta_t}{\varepsilon(i, j, k)} \left\{ \frac{1}{\Delta_y} \left[ H_y^{\bar{n}}(i, j, k) - H_y^n(i, j, k - 1) \right] - \frac{1}{\Delta_z} \left[ H_y^n(i, j, k) - H_y^n(i, j, k - 1) \right] \right\},
$$

(6.22)

$$
H_x^n(i, j, k) = H_x^{n-1}(i, j, k) + \frac{\Delta_t}{\mu(i, j, k)} \left\{ \frac{1}{\Delta_z} \left[ E_y^n(i, j, k) - E_y^n(i, j, k - 1) \right] - \frac{1}{\Delta_y} \left[ E_y^n(i, j, k) - E_y^n(i, j - 1, k) \right] \right\}.
$$

(6.23)
The update equations for the other field components can be found from these equations using symmetry. Equations (6.22) and (6.23) clearly indicate that each field component depends on the values of the four nearest field samples.

When the domain of interest is surrounded by free space with permittivity $\varepsilon_0$ and permeability $\mu_0$, it is truncated at a boundary $S_e$ that coincides with the faces of the Yee cells. Hence, the tangential electric field values on $S_e$, $E_{BC}$, cannot be updated using the standard FDTD equations as that would require the knowledge of the magnetic field components that exist outside the computational domain. Usually special mesh truncation schemes, such as those presented in [20-23], are used to update these fields. Such schemes are local in both space and time. Alternatively, one could enclose the body that is being analyzed by a hypothetical surface $S_x$ that lies inside the computational domain, compute the equivalent electric and magnetic currents $J(r,t)$ and $M(r,t)$ on this surface, and then compute the electric field at all points on $S_e$ using an RTBI. However, it is well known [76] that imposing the boundary condition on only the electric field on $S_e$ supports spurious modes that are excited by numerical errors. This can be overcome by supplementing the boundary condition on $S_e$ with a similar condition on tangential magnetic fields $H_{BC}$ on a surface $S_h$ that lies just inside $S_e$. Denoting the electric and magnetic fields computed using the equivalent currents by $E(r,t)\{J,M\}$ and $H(r,t)\{J,M\}$, the boundary conditions can be stated as

$$E_{BC} = E(r,t)\{J,M\} \text{ on } S_e,$$

$$H_{BC} = \frac{1}{2}(H_{FDTD} + H(r,t)\{J,M\}) \text{ on } S_h,$$

where $H_{FDTD}$ denotes the magnetic field updated using the standard update equations that involve $E_{BC}$. Details of this technique will be outlined next.
The electric field on the surface $S_e$ and the magnetic field on the surface $S_h$ due to the equivalent sources residing on $S_s$ are computed using a pair of RTBIs. To develop these equations, we begin by defining the equivalent currents on $S_s$ as $J(r, t) = \hat{n} \times \mathbf{H}(r, t)$ and $\mathbf{M}(r, t) = \mathbf{E}(r, t) \times \hat{n}$, where $\hat{n}$ is the unit normal to $S_s$. Then, it is easily verified that the magnetic and electric vector potentials due to these equivalent sources are given by

$$A(r, t) = \frac{\mu_0}{4\pi} \int_{S_s} dr' \frac{J(r', t) \cdot -R/c}{R}, \quad (6.26)$$

$$\mathbf{F}(r, t) = \frac{\varepsilon_0}{4\pi} \int_{S_s} dr' \frac{\mathbf{M}(r', t) \cdot -R/c}{R}, \quad (6.27)$$

where $R = |\mathbf{r}| = |\mathbf{r} - \mathbf{r}'|$. Once these potentials are known, the electric and magnetic fields may be computed using

$$\mathbf{E}(r, t)\{J, M\} = \int_0^t dt' \left( \partial_{r'}^2 \mathbf{I} - c^2 \nabla \nabla \right) \cdot \mathbf{A}(r, t') - \frac{1}{\varepsilon_0} \nabla \times \mathbf{F}(r, t) \quad \mathbf{r} \in S_e, \quad (6.28)$$

$$\mathbf{H}(r, t)\{J, M\} = \int_0^t dt' \left( \partial_{r'}^2 \mathbf{I} - c^2 \nabla \nabla \right) \cdot \mathbf{F}(r, t') + \frac{1}{\mu_0} \nabla \times \mathbf{A}(r, t) \quad \mathbf{r} \in S_h, \quad (6.29)$$

where $\mathbf{I}$ denotes the identity dyad. Needless to say, this is only one of the formalisms that are available [14, 16-18]. One could have used the Kirchhoff integral formula, as was done in [17]. However, as will be clear shortly, Eqs. (6.28) and (6.29) prove more amenable to efficient incorporation of the PWTD algorithm. To compute the electric and magnetic fields using Eqs. (6.28) and (6.29), the equivalent currents are approximated using subdomain bases in both space and time. To this end, $J(r, t)$ and $\mathbf{M}(r, t)$ are represented in terms of rooftop basis functions [76] $f_m(r)$, $m = 1, \ldots, N_s$, with time varying coefficients $J_m(t)$ and $M_m(t)$, respectively.
Furthermore, the coefficients $J_m(t)$ and $M_m(t)$ are also expanded in terms of third-order Lagrange interpolants given in Eq. (2.15) $T_n(t), n = 1, \ldots, N_t$. Thus, the currents on $S_x$ are represented using either of the representations

$$
J(r, t) = \sum_{m=1}^{N_x} J_m(t)f_m(r) = \sum_{m=1}^{N_x} \sum_{n=1}^{N_t} I_{m,n} T_n(t)f_m(r),
$$

(6.30)

$$
M(r, t) = \sum_{m=1}^{N_x} M_m(t)f_m(r) = \sum_{m=1}^{N_x} \sum_{n=1}^{N_t} K_{m,n} T_n(t)f_m(r),
$$

(6.31)

where $I_{m,n}$ and $K_{m,n}$ can be interpreted as expansion coefficients associated with the space-time basis function $T_n(t)f_m(r)$. Substituting these currents in Eqs. (6.26)-(6.29), the $i$ component of the fields at $r_{m'}$ and at $t = t_{n'} = n'\Delta_t$ can be expressed in matrix form as

$$
\Psi_{n'} = \sum_{n=1}^{n'-1} Z_n I_{n'-n},
$$

(6.32)

where

$$
\Psi_{n',m'} = \begin{cases} 
\hat{i} \cdot E(r_{m'}, t_{n'})\{J, M\} & \text{if } r_{m'} \in S_e \\
\hat{i} \cdot H(r_{m'}, t_{n'})\{J, M\} & \text{if } r_{m'} \in S_h,
\end{cases}
$$

(6.33)

$$
Z_n = \begin{bmatrix}
\bar{Z}_{n}^{EJ} & \bar{Z}_{n}^{EM} \\
\bar{Z}_{n}^{HJ} & \bar{Z}_{n}^{HM}
\end{bmatrix},
$$

(6.34)
\begin{align}
\overline{Z}_{n,m'}^E &= -\mathbf{t} \cdot \int_0^t dt' \left( \frac{\partial^2 \overline{\mathbf{E}}}{\partial t'^2} - c^2 \nabla \nabla \right) \cdot \mathbf{A}_m(r, t') \bigg|_{t=t', r=r'} , \\
\overline{Z}_{n,m'}^M &= \mathbf{t} \cdot \nabla \times \mathbf{F}_m(r, t') \bigg|_{t=t', r=r'} , \\
\overline{Z}_{n,m'}^{HJ} &= \mathbf{t} \cdot \nabla \times \mathbf{A}_m(r, t') \bigg|_{t=t', r=r'} , \\
\overline{Z}_{n,m'}^{HM} &= -\mathbf{t} \cdot \int_0^t dt' \left( \frac{\partial^2 \overline{\mathbf{E}}}{\partial t'^2} - c^2 \nabla \nabla \right) \cdot \mathbf{F}_m(r, t') \bigg|_{t=t', r=r'} .
\end{align}

(6.35)

In the above equations, \( \mathbf{A}_m(r, t) \) and \( \mathbf{F}_m(r, t) \) are the magnetic and electric vector potentials due to \( \mathbf{J}_m(r, t) = J_m(t) \mathbf{f}_m(r) \) and \( \mathbf{M}_m(r, t) = M_m(t) \mathbf{f}_m(r) \), respectively. It is now apparent that if these matrices are precomputed and stored, the electric and magnetic fields may be computed at the desired locations at every time step. This procedure is expensive, and the cost for computing the fields at the boundaries \( S_e \) and \( S_h \) scales as \( O(N_e N_h^2) \) since the number of observation points on the boundaries scales as \( O(N_h) \). This prohibitive cost is the main drawback of what would have been an ideal method of truncating the FDTD computational domain. The high computational complexity of evaluating fields at the boundary can be overcome by using the multilevel implementation of the PWTD algorithm that was explained in detail in Section 3.4.2. In what follows, how the algorithm of Section 3.4.2 can be adapted to the present application will be elucidated. Algorithmic details will be kept to a minimum while implementation details that are specific to this application will be discussed in depth.

### 6.4.2 Efficient evaluation of boundary fields using the PWTD algorithm

As was demonstrated in the previous chapters, efficient evaluation of fields due to a set of sources hinges upon expressing the former in terms of propagating plane waves. Hence, the first task of this section is the development of such plane wave expansions for the fields of Eqs. (6.28) and (6.29). Fortunately, this task proves to be a straightforward extension of the ideas
presented in Section 5.2, since a careful comparison of these equations with Eqs. (5.4) and (5.5) reveals that

\[
\hat{\mathbf{r}} \cdot \mathbf{E}(r, t) \{ \mathbf{J}, \mathbf{M} \} = \hat{\mathbf{r}} \cdot \mathcal{L}_e \{ \mathbf{J}(r, t) \} - \eta_0^2 \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} \times \mathcal{L}_h \{ \mathbf{M}(r, t) \},
\]

(6.36)

\[
\hat{\mathbf{r}} \cdot \mathbf{H}(r, t) \{ \mathbf{J}, \mathbf{M} \} = \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} \times \mathcal{L}_h \{ \mathbf{J}(r, t) \} + \eta_0^{-2} \hat{\mathbf{r}} \cdot \mathcal{L}_e \{ \mathbf{M}(r, t) \},
\]

(6.37)

where \( \mathcal{L}_e \{ \cdot \} \) and \( \mathcal{L}_h \{ \cdot \} \) are operators given in Eqs. (5.4) and (5.5), \( \eta_0 = \sqrt{\mu_0 / \varepsilon_0} \), and \( \hat{\mathbf{r}} \) and \( \hat{\mathbf{n}} \) are unit vectors tangential and normal to either of the surfaces \( S_e \) or \( S_h \). The plane wave expansions for the \( \mathcal{L}_e \{ \cdot \} \) and \( \mathcal{L}_h \{ \cdot \} \) operators were derived in Section 5.2. The conditions under which these expansions hold will be recalled and expansions for Eqs. (6.28) and (6.29) stated next.

With reference to Figure 5.1, consider two equally sized cubical boxes that are sufficiently separated from each other and each of which can be enclosed in a sphere of radius \( R_s \). These boxes are dubbed the source and observation boxes, respectively. The objective is to compute the fields at discrete locations in the observation box due to a set of sources in the source box. In the context of the scheme described in the previous section, assume that the source box contains a set of sources that are represented in terms of basis functions, as in Eqs. (6.30) and (6.31). The collection of these basis functions will be referred to as group \( \gamma \) and the collection of the observer locations in the observation box as group \( \gamma' \). Assume that the time signatures of the current densities \( J_m(t) \) and \( M_m(t) \) for \( f_m(r) \in \gamma \) can be divided into \( L \) consecutive subsignals of duration \( T_s \) each comprising \( M_t \) temporal samples

\[
J_m(t) = \sum_{l=1}^{L} J_{m,l}(t) = \sum_{l=1}^{L} \sum_{n^*=(l-1)M_t+1}^{LM_t} I_{m,n^*} \psi(t-n^*\Delta_t),
\]

(6.38)
\[ M_m(t) = \sum_{l=1}^{L} M_{m,l}(t) = \sum_{l=1}^{L} \sum_{n=(l-1)M_r+1}^{LM_r} K_{m,n^*} \psi(t - n^* \Delta t), \quad (6.39) \]

where \( \psi(t) \) is an interpolant which is of finite duration and which is approximately bandlimited to \( \omega_s = \epsilon_1 \omega_{\text{max}} > \omega_{\text{max}} \) as given in Eq. (3.41). With this partitioning of signals, \( J_{m,l}(t) \) and \( M_{m,l}(t) \) not only vanish outside the interval \( t_{l1}^{\text{min}} < t \leq t_{l1}^{\text{max}} \) such that \( t_{l1}^{\text{max}} - t_{l1}^{\text{min}} = T_s \) but also can be assumed bandlimited to \( \omega_s \) with a controllable amount of error. Then, by virtue of the Eqs. (5.21) and (5.22) and the discussion in Section 5.2.2, provided that the closest distance between the two groups is greater than \( c T_s \), the fields in group \( \gamma' \) due to sources \( J_{m,l} = J_{m,l}(t) f_m(r) \) and \( M_{m,l} = M_{m,l}(t) f_m(r) \) in group \( \gamma \) can be written as a collection of propagating plane waves for \( t > t_{l1}^{\text{max}} \) as

\[ \hat{\mathbf{E}}(r_m', t) \{ J_{m,l}, M_{m,l} \} = \eta_0 \int_{t_{l1}^{\text{max}}}^{t} dt' \sum_{p=0}^{K} \sum_{q=-K}^{K} w_{pq} \mathcal{P}^{-}(\mathbf{k}_{pq}, \hat{\mathbf{e}}, t) \cdot \mathcal{T}(\mathbf{k}_{pq}, \mathbf{R}_{c, \gamma', t}) \]
\[ \times \left[ \mathcal{P}^{+}(\mathbf{k}_{pq}, m, t) \ast J_{m,l} + \eta_0 \mathcal{P}^{+}(\mathbf{k}_{pq}, m, t) \ast M_{m,l} \right], \quad (6.40) \]

\[ \hat{\mathbf{H}}(r_m', t) \{ J_{m,l}, M_{m,l} \} = \frac{1}{\eta_0} \int_{t_{l1}^{\text{max}}}^{t} dt' \sum_{p=0}^{K} \sum_{q=-K}^{K} w_{pq} \mathcal{P}^{-}(\mathbf{k}_{pq}, \hat{\mathbf{e}}, t) \cdot \mathcal{T}(\mathbf{k}_{pq}, \mathbf{R}_{c, \gamma', t}) \]
\[ \times \left[ \eta_0 \mathcal{P}^{+}(\mathbf{k}_{pq}, m, t) \ast J_{m,l} + \mathbf{k}_{pq} \times \mathcal{P}^{+}(\mathbf{k}_{pq}, m, t) \ast M_{m,l} \right], \quad (6.41) \]

where the translation function is defined as

\[ \mathcal{T}(\mathbf{k}, t) = \frac{\partial^3}{16\pi^2 c R_c} \sum_{\nu=0}^{K} (2\nu + 1) P_{\nu} \left( \frac{ct}{R_{c, \gamma'}} \right) P_{\nu} \left( \mathbf{k} \cdot \mathbf{R}_{c, \gamma'} \right) \quad (6.42) \]

and the projection operators are defined as

\[ \mathcal{P}^{+}(\mathbf{k}, m, t) = \left[ \int_{\mathcal{S}_r} d\mathbf{r}' \mathbf{k} \times f_m(\mathbf{r}) \delta \left( t + \mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}_m') / c \right) \right], \quad (6.43) \]

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\[ \mathcal{P}^-(\hat{k}, \hat{i}, t) = \left[ i \delta \left( t - \frac{\hat{k} \cdot (r_{m'} - r_{\gamma'})}{c} \right) \right]^T \tag{6.44} \]

with a superscript \( T \) denotes a transpose, \( K = \left[ 2 \chi_1 \chi_2 R_s \omega_{\text{max}} / c \right] \) with an excess bandwidth factor \( \chi_2 > 1 \), \( P_\nu(\cdot) \) is the Legendre polynomial of degree \( \nu \), \( r_\gamma \) and \( r_{\gamma'} \) are the centers of the spheres enclosing groups \( \gamma \) and \( \gamma' \), and \( R_{c,\gamma\gamma'} \) and \( \hat{R}_{c,\gamma\gamma'} \) are the magnitude and direction of the vector \( R_{c,\gamma\gamma'} = r_\gamma - r_{\gamma'} \). The weights \( w_{pq} \) and plane wave directions \( \hat{k}_{pq} \) are as given in Eq. (5.30).

Equations (6.40) and (6.41) are the plane wave expansions for Eqs. (6.28) and (6.29), and can be implemented in a three-stage PWTD algorithm. Note that due to the nature of the \( \mathcal{P}^+(\hat{k}, m, t) \) projection operator, the electric and magnetic fields can be constructed using plane waves characterized by only two components perpendicular to the direction of its propagation. As natural as this seems, had we relied on the application of Kirchhoff integral formula to each component of the radiated field as was done in [17], three scalar plane waves would be needed in the PWTD algorithm. A second point to note is that there are two sources of errors in numerical evaluation of fields using Eqs. (6.40) and (6.41): (i) employing the subsignals in Eqs. (6.38) and (6.39) as if they were bandlimited and (ii) truncating the radiation pattern spectrum at \( K \) harmonics. However, it has been shown earlier that both these errors—and hence the error of the overall algorithm for evaluating the fields on the truncation boundaries—can be controlled to arbitrary precision by adjusting the parameters \( \chi_1 \) and \( \chi_2 \) and the temporal extent of the basis functions \( \psi(t) \). Finally, the constraint that the closest distance between the source and observer groups must be greater than \( R_{\text{min}} = cT_s \) implies that in a multiple box setting, one can classify group pairs based on whether they satisfy this separation criterion or not. The group pairs that are separated by more than this distance are classified as far-field pairs and the other pairs as near-field pairs. As usual, the interactions between the far-field pairs can be evaluated using the PWTD algorithm (Eqs. (6.40) and (6.41)), and Eqs. (6.28) and (6.29) can be used in evaluating
the near-field interactions. Furthermore, this scheme can be cast into a multilevel structure as briefly outlined next to yield a very efficient way of evaluating the fields on the truncation boundaries.

The first task in such a multilevel algorithm is to divide the scatterer into subscatterers and tabulate the near- and far-field interaction pairs in a hierarchical manner. To this end, the scatterer is enclosed in a fictitious cubical box that is recursively divided $N_I$ times. At any level, a parent box contains eight child boxes. All nonempty boxes at all levels are identified and a fundamental signal duration $T_S(\cdot)$ is associated with each level. Then at each level $\nu$, it is determined if a box pair is a far-field pair using the following argument: two boxes form a far-field pair if the distance between their closest points is greater than a prescribed distance $R_{\text{min}}(\nu) = cT_S(\nu)$, and if the distance between the closest points on their parents is less than $R_{\text{min}}(\nu + 1)$. Then, all far-field pairs at all levels are tabulated. At the lowest level, box pairs that have not been identified as a far-field pair are called near-field pairs. With this structure, the fields on the truncation boundaries are calculated via the following steps.

1. At each time step, all the equivalent source values on $S_s$ are computed and stored, and all near-field contributions to $\mathbf{\hat{t} \cdot E}(r_{m'}, t_{n'})\{J, M\}$ and $\mathbf{\hat{t} \cdot H}(r_{m'}, t_{n'})\{J, M\}$ for all $r_{m'} \in \{S_e, S_h\}$ are evaluated via Eqs. (6.28) and (6.29), respectively. Due to the local nature of the operations in this step, it can be completed in $O(N_s)$ operations at each time step.

2. When the current time step is an integer multiple of $T_S(\nu)/\Delta t$ for some level $\nu$, the outgoing rays for all groups at that level are computed. For the lowest level, this is done directly by evaluating the square bracketed term that appears in Eqs. (6.40) and (6.41). For higher levels, the outgoing rays are formed from the rays of one lower level through interpolation and resection operations described in Section 3.4.2. The most costly operation in this step is the interpolation operation, which can be performed in $O(N_s \log N_s)$ operations per level per
time step using the fast spherical filter of Jacob-Chien and Alpert [102]. As there exist $O(\log N_s)$ levels, the average cost of this step is $O(N_s \log^2 N_s)$ per time step.

3. Once the outgoing rays are formed at a level, translation of these rays, i.e., convolution of these rays with the translation functions, can be performed between groups that interact at that level. Translation operation can be performed most efficiently by first evaluating the Fourier transform of each outgoing ray using fast Fourier transforms (FFTs), then multiplying the resulting outgoing ray spectrum by the spectrum of the translation function, and finally inverse Fourier transforming the result. The resulting rays are superimposed on the incoming rays of the observation groups. Performing the FFTs proves to be the most costly operation in each ray translation, yielding an overall computational complexity of $O(N_s \log^2 N_s)$ per time step for this step. The cost of this step can be reduced without affecting the complexity by realizing that two real components are associated with each ray, and all the operations can be performed by using complex FFTs by combining the two real components into a single complex number [105].

4. When the current time step is an integer multiple of $T_s(\nu)/\Delta t_i$ for some level $\nu$, the incoming rays at that level are obtained by resecting and anterpolating the incoming rays at a higher level as described in Section 3.4.2. Finally, the far-field contributions $i \cdot E(r_{m'}, t_{n'})\{J, M\}$ and $i \cdot H(r_{m'}, t_{n'})\{J, M\}$ are evaluated at every time step by delaying and weighing the incoming rays at level 1 as dictated by the projection operators $P^{-}(\hat{k}_{pq}, i, t)\ast$. This step, which is conceptually the transpose of forming the outgoing rays, can also be accomplished in $O(N_s \log^2 N_s)$ operations.

Summing up the cost of each step, it is seen that the computational cost of this algorithm, which evaluates $i \cdot E(r_{m'}, t_{n'})\{J, M\}$ and $i \cdot H(r_{m'}, t_{n'})\{J, M\}$ for all $r_{m'} \in \{S_e, S_h\}$ for $N_t$ time steps, scales as $O(N_t N_s \log^2 N_s)$. This computational complexity is significantly lower than the
$O(N_tN_s^2)$ complexity of the classical algorithms. As shown in Section 5.4.1, the memory requirements of this algorithm scale as $O(N_tN_s)$ as compared to $O(N_tN_s^2)$ memory scaling of classical algorithms that avoid calculating the (direct) interaction coefficients at each time step. As will be demonstrated next, this efficient utilization of computational resources together with the fact that the truncation boundaries in this scheme can be chosen arbitrarily makes the proposed EBC the method of choice for analyzing wave phenomena in the presence of thin concave structures.

### 6.4.3 Numerical results

In this section, accuracy of the proposed EBC is inspected and its predicted computational complexity verified. The results obtained using EBCs are compared to those obtained using PML [23, 24]. The parameters for the PMLs will be indicated with the notation PML(a,b,c), where the entries a, b, and c denote the number of layers, the maximum conductivity, and the variation of the conductivity (2 for quadratic, 3 for cubic, etc.), respectively. In all the examples that follow, the FDTD cells are chosen to be cubic with edge length of $\Delta = \Delta_x = \Delta_y = \Delta_z = 1.25$ mm.

In the first numerical experiment, a transparent $+\hat{z}$ directed electric dipole is placed at the center of a cubical computational domain that measures 20 cells on each side. The temporal signature of the source is chosen to be a Gaussian pulse defined as

$$f(t) = e^{-(t-5\sigma)^2/2\sigma^2},$$

(6.45)

where $\sigma = 57.2 \times 10^{-12}$ s. The time step size is chosen to be $\Delta_t = 2.287 \times 10^{-12}$ s. This computational domain is truncated by the EBC and the total energy in it is calculated every time step. The time evolution of energy, normalized to its peak value, is shown in Figure 6.11 with the label EBC1. Had the mesh truncation been perfect, the energy inside the domain would
become negligible after a certain time. However, it is seen that some energy is left inside the computational domain, although at a very low level. To see the tradeoff between the computational requirements and accuracy, the fields on the integration boundary were represented using rooftop basis functions with a $4\Delta \times 2\Delta$ support, and the field values at only every other point on the truncation surfaces are evaluated and the rest of the field values are found by linear interpolation. The energy-versus-time diagram obtained with these choices is also plotted in Figure 6.11 as EBC2. As expected, slightly more energy remains in the computational domain. In order to assess the effectiveness of using the EBC, the same test was repeated with two different PMLs wrapped around the computational domain. The parameters of these PMLs were adjusted to yield minimum energy after time step 325 with a given number of layers. It is seen that the EBC1 performs as well as an eight-layer PML and EBC2 as well as a six-layer PML. The eight-layer PML and the EBC with fewer samples (EBC2) will be used for comparison purposes in the examples to follow.

![Graph showing energy versus time for different boundary conditions.]

**Figure 6.11:** Normalized energy in a $20 \times 20 \times 20$ cell computational domain enclosing an electric dipole and terminated with EBC and PMLs.

In the second example, the claim that the EBC can be imposed on arbitrary truncation boundaries is tested by analyzing transient scattering from an L-shaped PEC plate depicted in
Figure 6.12. The arms of the plate are 4 cells wide and 12 cells long each. This structure is illuminated by a Gaussian plane wave propagating in the $\hat{k}^{inc} = (-\hat{y} - \hat{z})/\sqrt{2}$ direction whose electric field is given by $E(r,t) = (-\hat{y} + \hat{z}) f(t - \hat{k}^{inc} \cdot r c)/\sqrt{2}$. Here, the function $f(\cdot)$ is given in Eq. (6.45) with $\sigma = 57.2 \times 10^{-12}$ s and the time step size was as before. First, the scatterer is enclosed in a $16 \times 22 \times 22$ cell domain which was wrapped in PML(8,6,2,2) and the $\hat{z}$ component of the electric field values throughout the domain were recorded at each time step. The incident field for this case was imposed using a total field/scattered field formulation [11]. Then, the EBC was imposed on an L-shaped boundary as shown in Figure 6.12 and the calculations repeated. This time, the incident field was imposed by adding it to the field values on the truncation boundaries. Snapshots of the scattered electric field calculated by both methods are shown in Figure 6.13. Clearly, the field distributions look identical even around the intruding part of the truncation surface for the EBC.
Figure 6.13: Scattered pressure field distribution around the L-shaped PEC plate at $t = 114 \times 10^{-12}$ s calculated using (a) PML(8,6,2,2) and (b) EBC.

In the previous two cases, due to the small sizes of the computational domains, not many field values were evaluated via the PWTD algorithm. A third test was devised to quantify the difference between using a 8-layer PML and imposing the EBC on a concave boundary on which most of the fields are calculated via the PWTD algorithm. To this end, again scattering from an L-shaped dielectric was studied. With reference to Figure 6.14, the dimensions and the properties of the scatterer were chosen as $a = 16\Delta$, $b = 4\Delta$, $\varepsilon = 4\varepsilon_0$, and $\mu = \mu_0$. The incident pulse was chosen as in the previous example except it was $\mathbf{\hat{x}}$ polarized, $\sigma$ was set to $112 \times 10^{-12}$ s to insure a sufficiently dense sampling within the scatterer, and $\Delta_t = 1.84 \times 10^{-12}$ s to ensure stability and sufficient temporal resolution. For the PML case, the scatterer was enclosed in a $26 \times 46 \times 46$ cell domain, and for the EBC calculations a $26 \times 32 \times 32$ cell part of this domain was removed so that $S_s$ and $S_e$ lay three and five cells away from the structure in all directions, respectively. The values of the $\mathbf{\hat{x}}$ component of the scattered electric field at $(x, y, z) = (8\Delta, 30\Delta, 2\Delta)$ are plotted in Figure 6.15. The $L_2$ norm of the difference in the scattered field normalized to the $L_2$ norm of the total field calculated using PML is approximately $2.1 \times 10^{-3}$. 

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Finally, the computational complexity of the PWTD-enhanced EBC method is verified. This is accomplished by analyzing plane wave scattering from the L-shaped dielectric scatterers defined with $a$ values of $4\Delta$, $8\Delta$, $16\Delta$, $32\Delta$, and $64\Delta$. In each case $b = 4\Delta$ and $S_x$ is positioned 3 cells away from the scatterer in all directions. The CPU times and the memory used by imposing EBC without the PWTD, PWTD-enhanced EBC, and a 10-layered PML for an analysis of 500 time steps are plotted with respect to the surface area of the outer truncation boundary $S_x$ in Figure 6.16, and are labeled as DIR, PWTD, and PML, respectively. In addition, reference lines with slopes 1.0, 1.5, and 2.0 are also drawn in these log-log graphs.
Clearly, the PWTD algorithm has reduced the cost of imposing the EBC to scale almost linearly with $N_s$ per time step from $O(N_s^2)$. On the other hand, the CPU and memory requirements for the PML implementation seem to converge to the $O(N_s^{1.5})$ curve as expected. From Figure 6.16, it is predicted that the current implementation of PWTD-enhanced EBC will become more economical to use when $a > 1200 \Delta$.

![Graph](image1)

Figure 6.16: Scaling of CPU time and memory for the EBC, PWTD-enhanced EBC, and PML methods.

6.5 Conclusions

This chapter discussed the application of the nonwindowed multilevel PWTD algorithms in reducing the computational cost associated with imposing EBCs in the FDTD method. To this end, first, the most computationally intensive part of imposing an EBC was identified as the evaluation of RTBIs to update the field values at the truncation boundaries. Indeed, if calculated using classical methods, updating these $N_s$ boundary field values would require $O(N_s^2)$ operations per time step. Then, plane wave expansions that can be implemented as a three-stage algorithm to evaluate the boundary fields were derived. Finally, it was shown that embedding
these three-stage algorithms into a multilevel scheme would reduce the cost of imposing the EBC to $O(N_s \log^2 N_2)$ per time step. Numerical examples that demonstrate the accuracy of the proposed EBCs and that verify the predicted computational complexities were presented. Based on the numerical results, it is predicted that the PWTD-enhanced EBCs will prove to be more economical for analyzing wave-structure interactions when the structure of interest is slender, concave, and very large.
CHAPTER 7
SUMMARY AND FUTURE WORK

In this thesis, efficient and accurate numerical techniques for analyzing linear wave phenomena in the time domain are presented. Formulation of integral equations that do not support internal resonance modes and development of fast schemes for efficient evaluation of retarded-time boundary-integrals (RTBIs) are the key ingredients of the schemes introduced in this dissertation. Development of these ingredients and demonstration of their applicability are presented as follows.

In anticipation of efficient methods, whether currently available integral equations were capable of producing accurate results when used in the numerical analysis of large-scale surface scattering phenomena was explored in Chapter 2. More specifically, the possibility of internal resonances corrupting the numerical solutions to the Kirchhoff integral equation and its normal derivative in acoustics and to the EFIE and MFIE in electromagnetics was inspected. This proved to be an area that had attracted scarce interest since, in theory, internal modes cannot be excited in impenetrable objects by an external excitation, and even though they are plausible solutions to the integral equations characterizing the exterior problem they would have to be ruled out due to the initial conditions. However, it was shown elsewhere as well as in Chapter 2 that, in practice, these modes are excited by the inaccuracies associated with the numerical solution procedure. The filtering method suggested in the literature for suppressing these modes could not be applied in large-scale analysis as the incident fields would have significant frequency components at one or more of the internal resonance frequencies and application of a filter would deteriorate the accuracy of the analysis. The approach in Chapter 2 to avoid corruption of results by resonances was to formulate the problem in terms of integral equations that do not support resonant modes. With these integral equations, numerical inaccuracies would
not excite spurious resonances in the solution, and the accuracy of the results would be guaranteed. Leveraging off the vast body of research in the frequency domain, a linear combination of the Kirchhoff integral equation with its normal derivative and a linear combination of the time domain EFIE with MFIE were introduced as CFIEs for analyzing acoustic scattering from rigid bodies and electromagnetic scattering from PEC targets, respectively. Their internal resonance suppressing characteristics were alluded to using conservation of energy arguments and verified via numerical examples. The MOT schemes for solving the introduced CFIEs were also explained in detail in this chapter. These CFIEs along with the MOT formulations for solving them form an essential part of the scattering analyses carried out in Chapters 4 and 5.

Having surmounted the accuracy problem, techniques for overcoming the exorbitant computational requirements of using integral equation–based methods for analyzing transient scattering from large scatterers were developed in Chapter 3. To this end, a generic scalar MOT scheme was constructed and the evaluation of the associated RTBI was identified as the computational bottleneck in the implementation of this scheme. Then, a three-stage PWTD algorithm based on a Whittaker-type expansion of radiated wave fields was derived. It was shown that, subject to some spatial and temporal constraints, this PWTD algorithm could be used in place of the RTBI to evaluate the fields due to a set of sources with a desired level of accuracy. The important feature of the introduced PWTD algorithm was that it related the fields in a region of space to their sources that are confined in space and time by a diagonal operator, whereas the RTBI used a full matrix to express the same relation. Hence, the RTBI could be efficiently evaluated by arranging source and field observation points into groups and using the PWTD algorithm (instead of directly applying the RTBI) to find the fields in a group due to the sources in another one if the space-time constraints were satisfied. Using groups of equal sizes resulted in a two-level algorithm with a computational complexity of $O(N_r N_s^{1.5} \log N_s)$ when
evaluating fields for \( N_t \) time steps at \( N_s \) points distributed on the same surface that \( N_s \) sources reside on. The same results could be obtained with the direct evaluation of the RTBI in \( O(N_t N_s^2) \) operations. In passing, it should be mentioned that the number of sources or observers in a group has to scale as \( O(N_s^{0.5}) \) to assure the mentioned complexity of the two-level scheme. To arrive at a lower complexity scheme, the PWTD algorithm was cast in a multilevel framework where smaller (lower level) groups were aggregated together to form larger groups that interact more efficiently through the PWTD algorithm. Accomplishing efficient exchange of information between different levels was crucial for the success of this scheme. The cost scaling of the resulting multilevel scheme was derived to be \( O(N_t N_s \log^2 N_s) \)—significantly lower than that of the direct method. The PWTD algorithm thus far used plane waves that propagate in all directions; even those that originated from the source domain and propagated away from the observation region. This was necessitated by the use of a truncated translation operator that enabled the use of a small number of plane wave directions in numerically evaluating a spherical spectral integral. In the second part of Chapter 3, how the number of plane wave directions used in implementation could be reduced was inspected. To this end, the radiated fields were expanded using a finite-cone representation consisting of a limited number of plane waves whose propagation directions formed a cone encompassing the observation domain. The key to the success of this expansion was truncating the spectral content of the translation function by using a smooth window so that it spanned a shorter duration. Hence, the resulting three-stage scheme was called the windowed PWTD algorithm. It was shown that the use of this algorithm in two-level and multilevel settings reduced the cost of evaluating an RTBI to \( O(N_t N_s^{4/3} \log N_s) \) and \( O(N_t N_s \log N_s) \), respectively.

Chapters 4 to 6 detailed the incorporation of the nonwindowed PWTD schemes into time domain acoustic and electromagnetic simulation codes. Chapter 4 built upon the MOT scheme
introduced in Chapter 2 to solve the acoustic CFIE by first identifying the RTBI associated with this scheme and by providing a three-stage PWTD algorithm to replicate its effect. Implementation details in casting this three-stage process into two-level and multilevel schemes were also elucidated in this chapter. The efficacy and predicted computational complexity of the two-level and multilevel schemes were demonstrated through several numerical examples. In particular, it was seen that using the PWTD-enhanced algorithms becomes more efficient than using the classical MOT solvers for problems with as little as 1600 spatial unknowns. With the new schemes, problems with $N_s > 100000$ could be handled in the same computer environment that could only accommodate problems with $N_s < 7500$ with the classical methods. Chapter 5 followed the same trails as Chapter 4 to lower the computational burden associated with the MOT scheme for solving the electromagnetic CFIE also introduced in Chapter 2. This time, however, properties associated with the vector nature of the problem were also used in increasing the efficiency of the resulting codes without affecting their computational complexities. Due to this feature, the breakeven points between the PWTD-enhanced methods and the classical ones were found to be around $N_s = 1000$—lower than that for the acoustic case. The largest electromagnetic case studied also involved in excess of 100 000 spatial unknowns. Hence, these two chapters show that the goal of developing efficient codes for analyzing wave interactions with realistic structures has been accomplished.

Chapter 6 presented the use of the PWTD technology in enhancing the performance of differential equation–based methods. However, the use of the PWTD algorithm is not much different from what was presented in the previous chapters in the sense that it is again used in rapid evaluation of RTBIs. This time, the RTBIs are used in evaluating the fields that lie on the truncation boundaries of an FDTD domain. Since the field values external to the truncation boundary are not known, these field values cannot be updated using the standard FDTD equations. The so-called exact boundary condition (EBC) method places an integration
boundary just inside the truncation boundary and expresses the fields on the latter one in terms of RTBIs of the fields on the former. Denoting the number of field samples on the truncation boundary as $N_s$ and imposing the EBC using classical methods for $N_t$ time steps requires $O(N_t N_s^2)$ operations, whereas the cost of imposing local grid truncation methods scales only as $O(N_t N_s)$. Hence, in a general case imposing the EBC proves to be even more costly than updating the $O(N_s^{1.5})$ field quantities within the computational domain every time step. Although using the multilevel PWTD algorithm, the EBC cost scaling can be reduced to $O(N_t N_s \log^2 N_s)$—lower than that of field updates—it still proves to be more expensive than using local grid truncation schemes. On the other hand, if the structure under consideration is slender, i.e., its volume scales as its surface area, and concave, using PWTD-enhanced EBCs seems worthwhile. This is because the local boundary conditions cannot be imposed on concave surfaces intruding inward to the computational domain. In Chapter 6, this prediction was verified both for the electromagnetic and acoustic cases by formulating PWTD schemes to reduce the cost of imposing EBCs and by comparing the complexities of the resulting algorithms against those of using perfectly matched layers for grid truncation. Although it was shown that for slender concave structures the complexity of using PWTD-enhanced EBCs was lower than that of using local mesh truncation schemes, the breakeven point was still quite high with the current implementation. Imposing the EBCs on two truncation surfaces to avoid internal resonance problems and the mismatch of field sampling densities required by the FDTD and RTBI discretizations can be stated as the main reasons for this high breakeven point. This immediately suggests further research into implementing PWTD-enhanced EBCs for truncating time domain finite element meshes. As has been verified in the frequency domain literature, such a hybridization provides a better match between the differential equation and integral equation–based parts of the resulting method. On the other hand, both the FDTD-based and finite element–based codes can be easily extended to efficiently handle wave interactions with
multiple separated objects as suggested in [119]. Among other advantages, increasing efficiency of multilevel PWTD schemes with increasing distance between subscatterers should prove invaluable in such applications.

As the basic PWTD algorithm provides an efficient way to evaluate fields that can be expressed as a convolution of the sources with the free-space Green’s function (and spatial and temporal derivatives thereof), it should find a vast area of applications. One immediate extension to the applications in Chapters 4 and 5 is use of the PWTD technology in surface scattering analysis from penetrable homogeneous structures. The complexities of classical methods for performing such analyses scale in proportion to those for analyzing scattering from impenetrable structures. Therefore, a reduction of cost proportional to that achieved in Chapters 4 and 5 should also be expected. Another application for the PWTD algorithms is their incorporation into time domain volume integral equation solvers. Since the source and observer points are distributed throughout a volume rather than over a surface in these solvers, greater speed-ups seem achievable through the use of PWTD technology. Finally, research should be directed towards expanding the scope of the basic PWTD algorithm from free-space kernels to kernels for lossy, dispersive, and layered media.
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VITA

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